Fourier transform applications: Optics and waves

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ELEC8501: The Fourier Transform and Its Applications

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https://www.eee.hku.hk/~elec8501

Preamble

Fourier transform is widely used in the analysis of wave propagation, such as in optics:

1. waves are sinusoidal, so decomposition of a signal in different "harmonics" makes sense
2. we often have linear space-invariant systems, where Fourier analysis is an effective tool

In this lecture, we consider in particular of image formation from the perspective of Fourier optics, but similar principle is applicable for optical communication, acoustic waves, etc.

1 Wave propagation

2 Fraunhofer and the use of lenses

3 Optical image formation

Diffraction and resolution

Motivation question: If we have a "point" object, is its image a "point"?

Diffraction and resolution

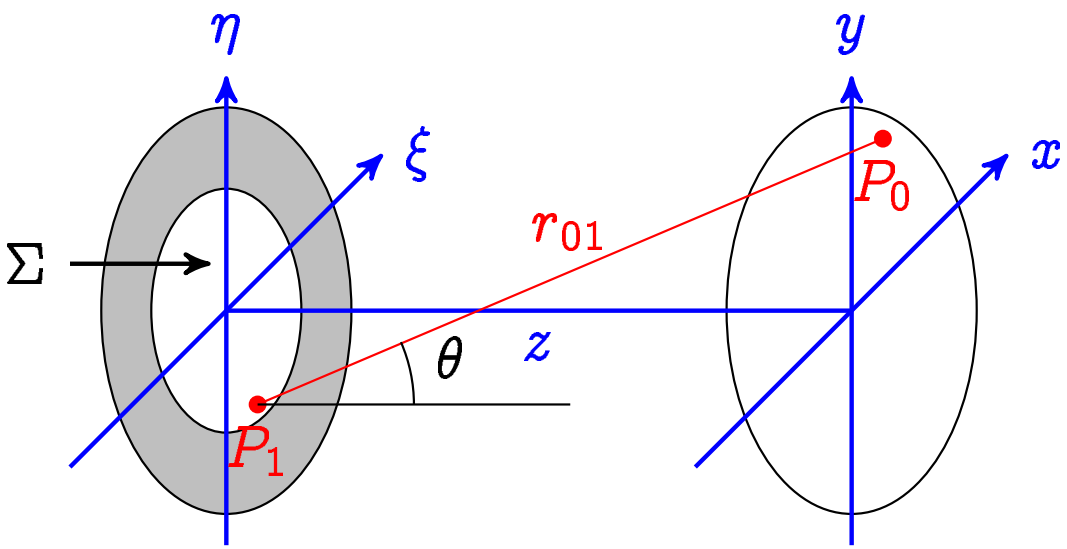
Motivation question: If we have a "point" object, is its image a "point"?

- What is a point? $\delta(x, y)$
- What is the implication that the image is a point? "infinite resolution". What is "resolution"?
- The physical phenomenon of diffraction

The discussions below follow the presentation in J.W. Goodman, *Fourier Optics*, 4th ed. W.H. Freeman, 2017.

Huygens-Fresnel principle

A general setup, with Σ as the diffracting aperture, and wavelength λ and wavenumber $k = 2\pi/\lambda$:



The complex phasor representation of the light field is given by

$$U(x,y) = \frac{z}{j\lambda} \iint_{\Sigma} U(\xi,\eta) \frac{e^{jkr_{01}}}{r_{01}^2} \, d\xi \, d\eta \tag{1}$$

Huygens-Fresnel principle

Approximations: Let us consider r_{01}.

$$r_{01} = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2} \tag{2}$$

$$= z\left[1 + \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2\right]^{1/2} \tag{3}$$

$$\approx z\left[1 + \frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right] \tag{4}$$

because of the binomial expansion $\sqrt{1+b} = 1 + \frac{1}{2}b - \frac{1}{8}b^2 + \ldots$.

$$r_{01} \approx z \qquad \text{(in the denominator)}$$

$$r_{01} \approx z\left[1 + \frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right] \qquad \text{(in the exponential)} \tag{5}$$

Huygens-Fresnel principle

Putting this in, and define $U(\xi, \eta, 0)$ to incorporate Σ:

$$U(x, y, z) = \frac{e^{jkz}}{j\lambda z} \iint_{-\infty}^{\infty} U(\xi, \eta, 0) e^{j\frac{k}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right]} \, d\xi \, d\eta \tag{7}$$

What do you notice about this equation?

Huygens-Fresnel principle

Putting this in, and define $U(\xi, \eta, 0)$ to incorporate Σ:

$$\boxed{U(x,y,z) = \frac{e^{jkz}}{j\lambda z} \iint_{-\infty}^{\infty} U(\xi,\eta,0) e^{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]} \, d\xi \, d\eta} \tag{7}$$

What do you notice about this equation? A convolution with

$$h(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \tag{8}$$

$$\boxed{U(x,y,z) = \iint_{-\infty}^{\infty} U(\xi,\eta,0) h(x-\xi, y-\eta) \, d\xi \, d\eta} \tag{9}$$

Huygens-Fresnel principle

$$U(x,y,z) = \frac{e^{jkz}}{j\lambda z} \iint_{-\infty}^{\infty} U(\xi,\eta,0) e^{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]} \, d\xi \, d\eta \tag{10}$$

What do you notice about this equation?

Huygens-Fresnel principle

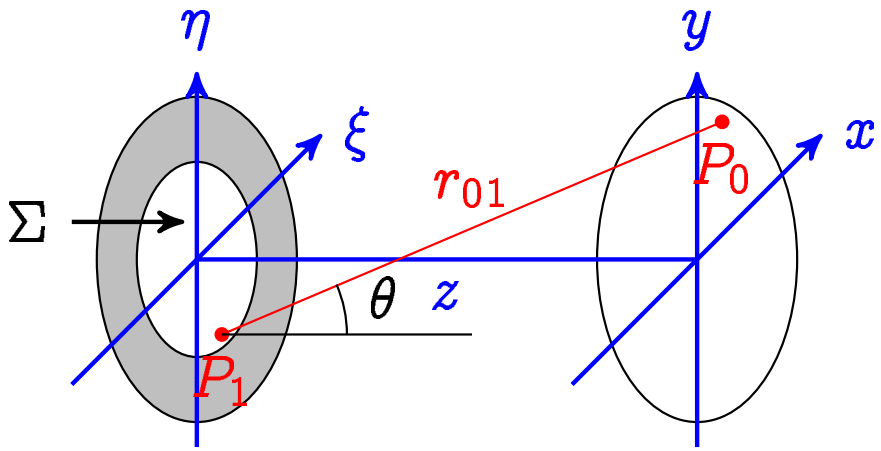
$$U(x,y,z) = \frac{e^{jkz}}{j\lambda z} \iint_{-\infty}^{\infty} U(\xi,\eta,0) e^{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]} \, d\xi \, d\eta \tag{10}$$

What do you notice about this equation? A Fourier transform! Let

$$C = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \tag{11}$$

$$U(x,y,z) = C \iint_{-\infty}^{\infty} \left[U(\xi,\eta,0) e^{j\frac{k}{2z}(\xi^2+\eta^2)}\right] e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)} \, d\xi \, d\eta \tag{12}$$

Huygens-Fresnel principle



What we have done is called *Fresnel approximation*. So, in the region of Fresnel diffraction, the equation

$$U(x,y,z) = C \iint_{-\infty}^{\infty} \left[U(\xi,\eta,0) e^{j\frac{k}{2z}(\xi^2+\eta^2)} \right] e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)} \, d\xi \, d\eta$$

says the complex field at a distance z is equal to the Fourier transform of the product of the complex field just after the aperture and a quadratic phase exponential.

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Fraunhofer approximation

How to get rid of the quadratic phase exponential?

Fraunhofer approximation

How to get rid of the quadratic phase exponential? Go really far.

If the distance z is far enough, known as the Fraunhofer (far-field) region, with

$$z \gg \frac{[k(\xi^2 + \eta^2)]_{\max}}{2} \tag{13}$$

then the Fraunhofer diffraction gives

$$\boxed{U(x, y, z) = C \iint_{-\infty}^{\infty} U(\xi, \eta, 0) e^{-j\frac{2\pi}{\lambda z}(x\xi + y\eta)} \, d\xi \, d\eta} \tag{14}$$

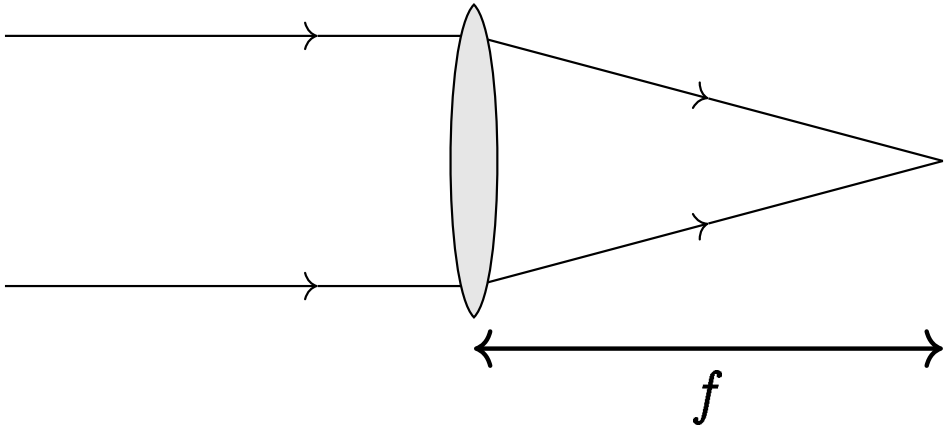
i.e., a Fourier transform of $U(\xi, \eta, 0)$!

What is a lens?

How to get rid of the quadratic phase exponential?

What is a lens?

How to get rid of the quadratic phase exponential? Use a lens.

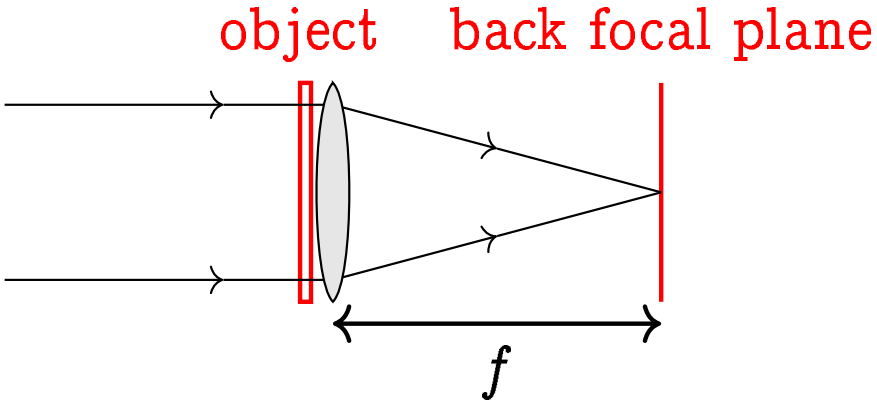


In simple terms, a convex lens (with positive f) multiplies

$$\boxed{U'_\ell(x,y) = e^{-j\frac{k}{2f}(x^2+y^2)}} \tag{15}$$

to the light field before it

Input placed against the lens



1. Object: $t_A(x, y)$
2. Disturbance incident on the lens: $U_\ell(x, y) = At_A(x, y)$
3. Lens has a pupil function $P(x, y)$ where

$$P(x, y) = \begin{cases} 1 & \text{inside the lens aperture} \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

4. Amplitude distribution behind the lens:

$$U'_\ell(x, y) = At_A(x, y)P(x, y)e^{-j\frac{k}{2f}(x^2+y^2)} \tag{17}$$

Input placed against the lens

Put this in Eq. (12), with $z = f$, and changing some notations:

$$U_f(u,v) = C \iint_{-\infty}^{\infty} \left[U'_\ell(x,y) e^{j\frac{k}{2f}(x^2+y^2)} \right] e^{-j\frac{2\pi}{\lambda f}(xu+yv)} \, dx \, dy \tag{18}$$

$$= C' \iint_{-\infty}^{\infty} t_A(x,y) P(x,y) e^{-j\frac{2\pi}{\lambda f}(xu+yv)} \, dx \, dy \tag{19}$$

If the physical extent of $P(x,y)$ is much larger than $t_A(x,y)$, it can be ignored, and we have the Fourier transform (or Fraunhofer diffraction pattern) of $t_A(x,y)$ at the back focal plane (rather than "far field")!

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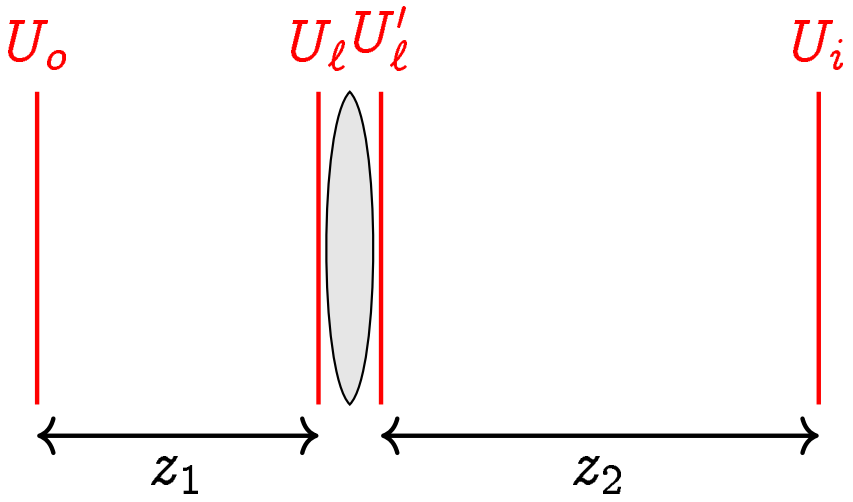
Geometry for image formation

More generally, we have the following setup:



Our goal is to find the *image* at U_i of a *point* at U_o.

- Input: $\delta(\xi, \eta)$
- At U_ℓ: a spherical wave diverging from the point (ξ, η):

$$U_\ell(x, y) = \frac{1}{j\lambda z_1} e^{j\frac{k}{2z_1}\left[(x-\xi)^2+(y-\eta)^2\right]} \tag{20}$$

Geometry for image formation

- Passing through the lens:

$$U'_\ell(x,y) = U_\ell(x,y)P(x,y)e^{-j\frac{k}{2f}(x^2+y^2)} \tag{21}$$

- Using the Fresnel diffraction equation given by Eq. (10):

$$h(u,v;\xi,\eta) = \frac{1}{j\lambda z_1}\iint_{-\infty}^{\infty} U'_\ell(x,y)e^{j\frac{k}{2z_2}\left[(u-x)^2+(v-y)^2\right]}\,dx\,dy \tag{22}$$

$$\begin{aligned} h(u,v;\xi,\eta) = {} & \frac{1}{\lambda^2 z_1 z_2}e^{j\frac{k}{2z_2}(u^2+v^2)}e^{j\frac{k}{2z_1}(\xi^2+\eta^2)} \\ & \times \iint_{-\infty}^{\infty} P(x,y)e^{j\frac{k}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}\right)(x^2+y^2)} \\ & \times e^{-jk\left[\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{v}{z_2}\right)y\right]}\,dx\,dy \end{aligned} \tag{23}$$

Simplifications

A series of simplifications:

1. Ignore the quadratic phase factors outside the integrations
2. Assume the object and image distances obey the lens formula:

$$\boxed{\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f}} \tag{24}$$

3. Define the magnification

$$\boxed{M = -\frac{z_2}{z_1}} \tag{25}$$

4. Scale the coordinates

$$\boxed{\hat{\xi} = M\xi \qquad \hat{\eta} = M\eta} \tag{26}$$

Impulse response

This "image of a point" becomes space-invariant (and let $z_i = z_2$ for image distance):

$$h(u-\hat{\xi}, v-\hat{\eta}) = K \iint_{-\infty}^{\infty} P(x,y) e^{-j\frac{2\pi}{\lambda z_i}\left[(u-\hat{\xi})x+(v-\hat{\eta})y\right]} \, dx \, dy \tag{27}$$

$$\boxed{h(u,v) = K \iint_{-\infty}^{\infty} P(x,y) e^{-j\frac{2\pi}{\lambda z_i}(ux+vy)} \, dx \, dy} \tag{28}$$

For a diffraction-limited system we can regard the image as being a convolution of the image predicted by geometrical optics with an impulse response that is proportional to the Fraunhofer diffraction pattern of the exit pupil. (Introduction to Fourier Optics, p. 189.)

Frequency analysis

With an impulse response, we can now perform a frequency analysis of the optical imaging system. However, with optical systems, we often deal with polychromatic illumination, leading to two special cases: *coherent imaging* and *incoherent imaging*:

- Spatially coherent: the phasor amplitudes of the field at all object points vary in unison $\Longrightarrow$ Linear in complex amplitude
- Spatially incoherent: the phasor amplitudes at all points on the object are varying in totally uncorrelated fashions $\Longrightarrow$ Linear in intensity, and impulse response is the squared magnitude of the amplitude impulse response

Coherent imaging

Amplitude transfer function:

$$\boxed{H(f_x, f_y) = \iint_{-\infty}^{\infty} h(u,v) e^{-j2\pi(f_x u + f_y v)}\, du\, dv} \tag{29}$$

which leads to:

$$H(f_x, f_y) = P(\lambda z_i f_x, \lambda z_i f_y) \tag{30}$$

So the frequency response has the same form as the pupil function!

Incoherent imaging

Optical transfer function (OTF):

$$\mathcal{H}(f_x, f_y) = \frac{\iint_{-\infty}^{\infty} |h(u,v)|^2 e^{-j2\pi(f_x u + f_y v)}\, du\, dv}{\iint_{-\infty}^{\infty} |h(u,v)|^2\, du\, dv} \tag{31}$$

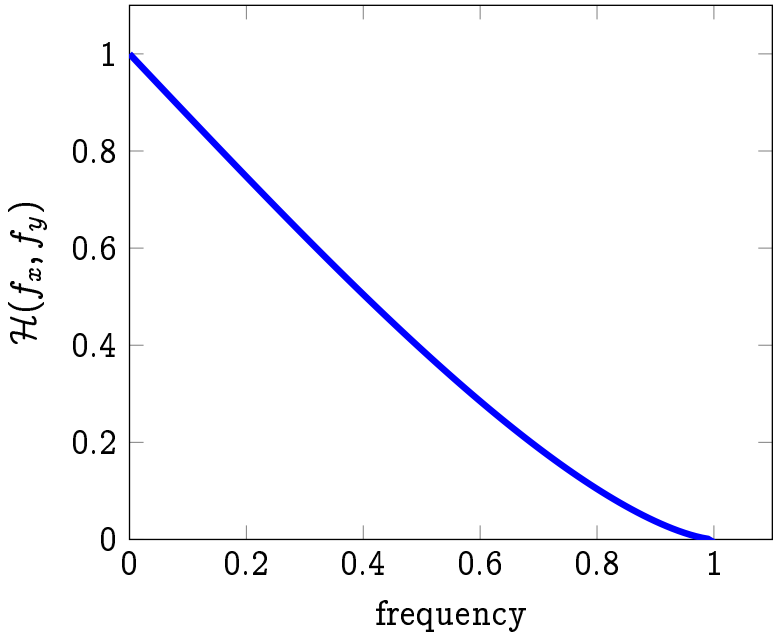
$$= \frac{\iint_{-\infty}^{\infty} H(p',q') H^*(p'-f_x, q'-f_y)\, dp'\, dq'}{\iint_{-\infty}^{\infty} |H(p',q')|^2\, dp'\, dq'} \tag{32}$$

OTF is the normalized autocorrelation function of the amplitude transfer function! (Introduction to Fourier Optics, p. 199.)

Incoherent imaging

A special case: $P(x, y)$ is a circ function.

- OTF is the autocorrelation of the circ function.



- There is attenuation of high frequency
- There is a cutoff frequency

The equations derived are general enough to incorporate other forms of optical aberrations

Resolution

The inverse Fourier transform of this OTF is jinc squared, also known as the Airy pattern.

- Each "point" becomes an Airy pattern in the image
- The image has a certain size, with cutoff at 1.22
- Rayleigh criterion of resolution: The center of the Airy intensity pattern generated by one point source falls exactly on the first zero of the Airy pattern generated by the second
- Often expressed as

$$\boxed{\delta x = 1.22 \frac{\lambda}{2n \sin\theta} = 0.61 \frac{\lambda}{\mathrm{NA}}} \tag{33}$$

where θ is the half angle subtended by the exit pupil and NA is called the numerical aperture.

Lithographic resolution

In lithography, you will often see this equation:†

$$\boxed{\mathrm{CD} = k_1 \frac{\lambda}{\mathrm{NA}}} \tag{34}$$

- CD = critical dimension: the smaller the better!
- k_1 = all process-related factors
- λ = wavelength of light source
- NA = range of angles of light

† J. Van Schoot, "The Moore's Law Machine," *IEEE Spectrum*, pp. 44–48, September 2023.

Lithographic resolution

- Most difficult to reduce the wavelength, but that has been done a few times:

  $365\,\text{nm} \rightarrow 248\,\text{nm} \rightarrow 193\,\text{nm} \rightarrow 13.5\,\text{nm}$ (Extreme Ultraviolet, EUV)

- When it was 193 nm, a trick to increase NA (from 0.93 to 1.35) was through immersion (called "immersion lithography").
- "Design for manufacturing" also helps to reduce k_1

Huge engineering efforts to build EUV systems, and now High-NA EUV systems!