

Fourier transform applications: Tomographic imaging

Dr. Edmund Lam

Department of Electrical and Electronic Engineering
The University of Hong Kong

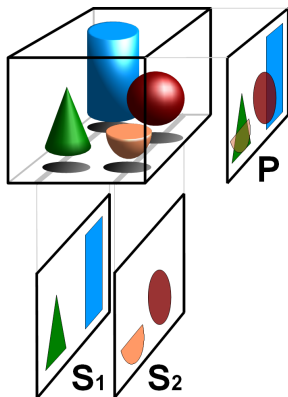
ELEC8501: The Fourier Transform and Its Applications
(First Semester, 2023–24)

<https://www.eee.hku.hk/~elec8501>

- 1 Tomography problem statement
- 2 Projection and slice
- 3 Circularly symmetric function
- 4 Filtered backprojection

Situation

A common problem, such as in medical imaging (computed tomography):



Source: Wikipedia entry on “Tomography”

Computed tomography

CT was invented by a music company (EMI) engineer, Godfrey Hounsfield[†]

- A chance conversation with a doctor while on vacation in the 1960s
- X-ray machine was invented in 1896; but X-ray images of the brain were too grainy and only two-dimensional
- Hounsfield had background in electronics and radar
- Came up with the idea to project narrow beams of X-rays through a person's head, and a computer would use the resulting data to construct a series of cross-sections; then teamed up with neuroradiologists
- EMI had some money for him (thanks to The Beatles record sales)
- *Co-recipient* of the 1979 Nobel Prize in Physiology or Medicine

[†] J. Goodrich, "The CT Scanner Was Invented by a Music Company Engineer," *IEEE Spectrum*, pp. 62–63, September 2023.

Computed tomography

- Another co-recipient: Allan Cormack
- For his development of the *computational algorithms* to reconstruct images from CT scans
- His Nobel Lecture (<https://www.nobelprize.org/uploads/2018/06/cormack-lecture.pdf>) entitled “Early Two-dimensional Reconstruction and Recent Topics Stemming from It”) mentions how this related to the work of Radon and Abel

Situation

For simplicity, let us fix at a certain vertical position (z).

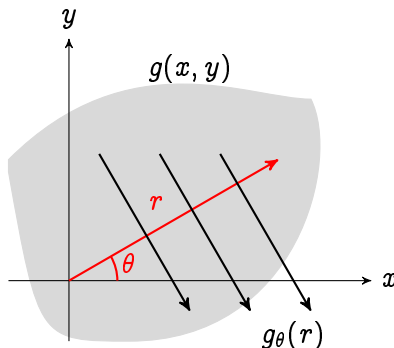
Questions:

- With “enough” (1D) projection at different angles, can we reconstruct the full (2D) signal?
- What is enough?
- What is the reconstruction algorithm?

Reconstruction from projections

To put it in mathematical terms:

Given $g_\theta(r)$ for a range of θ , reconstruct $g(x, y)$.



Reconstruction from projections

We have seen this setup in studying “line delta” in Lecture 3. Each line of projection is

$$r = x \cos \theta + y \sin \theta \quad (1)$$

So how about this:

- 1 Define

$$b_{\theta}(x, y) = g_{\theta}(x \cos \theta + y \sin \theta) \quad (2)$$

(this is a *backprojection* step)

- 2 Compute

$$g(x, y) = \int_0^{\pi} b_{\theta}(x, y) d\theta \quad (3)$$

(this is a *summation* step)

Turns out we can do better...with Fourier transform!

- 1 Tomography problem statement
- 2 **Projection and slice**
- 3 Circularly symmetric function
- 4 Filtered backprojection

Projection

Consider a vertical projection, i.e., $\theta = 0$.

$$g_0(x) = \int_{-\infty}^{\infty} g(x, y) dy \quad (4)$$

This has a (1D) Fourier transform:

$$\begin{aligned} g_0(x) &\supset \int_{-\infty}^{\infty} g_0(x) e^{-j2\pi ux} dx \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(x, y) dy \right] e^{-j2\pi ux} dx \\ &= G(u, 0) \end{aligned} \quad (5)$$

Projection

The very simple derivations above have a profound implication:

Projection of 2D $\xrightarrow{\mathcal{F}_{1D}}$ Slice of 2D in the Fourier domain

This is known as the projection-slice theorem, or Fourier-slice theorem.

But there shouldn't be anything special about $\theta = 0 \dots$ how about other angles?

Radon transform

The Radon transform of a 2D function (at angle θ) is given by

$$g_{\theta}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \delta(r - x \cos \theta - y \sin \theta) dx dy \quad (6)$$

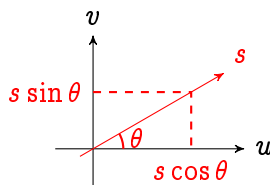
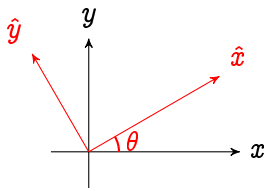
We want to show that $\mathcal{F}\{g_{\theta}(r)\} = \text{slice of 2D Fourier at angle } \theta$

Radon transform

The Radon transform of a 2D function (at angle θ) is given by

$$g_{\theta}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \delta(r - x \cos \theta - y \sin \theta) dx dy \quad (6)$$

We want to show that $\mathcal{F}\{g_{\theta}(r)\} = \text{slice of 2D Fourier at angle } \theta$



$$\hat{x} = x \cos \theta + y \sin \theta \quad (7)$$

$$\hat{y} = -x \sin \theta + y \cos \theta \quad (8)$$

Radon transform

$$g_{\theta}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \delta(r - x \cos \theta - y \sin \theta) dx dy \quad (9)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{g}(\hat{x}, \hat{y}) \delta(r - \hat{x}) d\hat{x} d\hat{y} \quad (10)$$

$$= \int_{-\infty}^{\infty} \hat{g}(r, \hat{y}) d\hat{y} \quad (11)$$

Then,

$$g_{\theta}(\hat{x}) \supset \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \hat{g}(\hat{x}, \hat{y}) d\hat{y} \right] e^{-j2\pi s \hat{x}} d\hat{x} \quad (12)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{g}(\hat{x}, \hat{y}) e^{-j2\pi s \hat{x}} d\hat{x} d\hat{y} \quad (13)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi s(x \cos \theta + y \sin \theta)} dx dy \quad (14)$$

$$= G(s \cos \theta, s \sin \theta) \quad (15)$$

Projection-slice theorem

We now have a more general statement:

Projection (at θ) of 2D $\xrightarrow{\mathcal{F}_{1D}}$ Slice of 2D (at θ) in Fourier

Let us now see how we can make use of this.

- 1 Tomography problem statement
- 2 Projection and slice
- 3 Circularly symmetric function**
- 4 Filtered backprojection

Circularly symmetric function

Consider a special case: what if $g(x, y)$ is circularly symmetric?

From Lecture 3: 2D Fourier transform of a circularly symmetric function is also circularly symmetric. We want to combine this fact with the Fourier-slice theorem to show that *one projection is enough*.

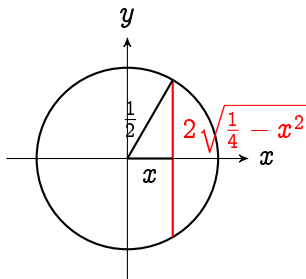
Abel transform

The projection of a circularly symmetric function $g(r)$ is given by the Abel transform:

$$g_A(x) = \mathcal{A}\{g(r)\} = \int_{-\infty}^{\infty} g\left(\sqrt{x^2 + y^2}\right) dy = 2 \int_x^{\infty} \frac{g(r)}{\sqrt{r^2 - x^2}} r dr \quad (16)$$

since $r = \sqrt{x^2 + y^2}$ gives $\frac{dr}{dy} = \frac{\sqrt{r^2 - x^2}}{r}$.

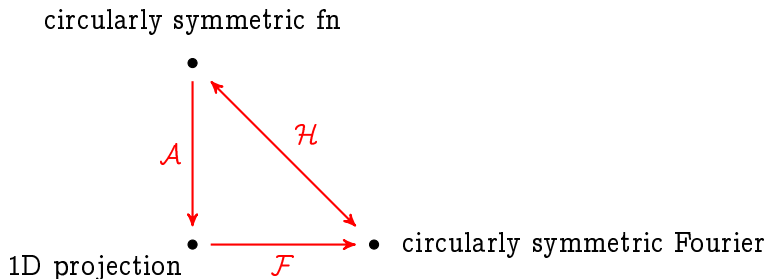
Example:



$$\mathcal{A}\{\Pi(r)\} = \sqrt{1 - 4x^2}\Pi(x)$$

Abel-Hankel-Fourier ring

Projection-slice theorem says: A circularly symmetric function is related to its “projection (Abel) then Fourier” by a Hankel transform



$$\mathcal{H} = \mathcal{F}\mathcal{A}$$

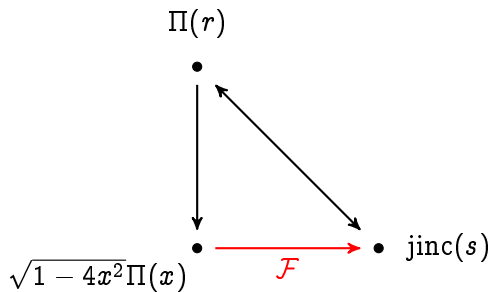
(17)

$$\mathcal{H}\mathcal{F}\mathcal{A} = \text{identity}$$

(18)

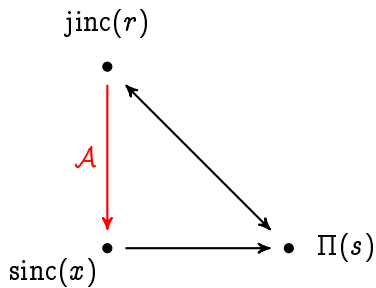
Abel-Hankel-Fourier ring

Examples:



Abel-Hankel-Fourier ring

Examples:



- 1 Tomography problem statement
- 2 Projection and slice
- 3 Circularly symmetric function
- 4 Filtered backprojection**

Non-symmetric function

For functions that are not necessarily circularly symmetric, one projection is not enough. We will need to take projections at different angles. But how do we combine them?

Remember that we have data $G_\theta(s) \triangleq G(s \cos \theta, s \sin \theta)$.

Tomographic reconstruction

From the inverse Fourier transform formula:

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u, v) e^{j2\pi(xu + yv)} du dv \quad (19)$$

$$= \int_0^{2\pi} \int_0^{\infty} G_{\theta}(s) e^{j2\pi(s \cos \theta \cdot x + s \sin \theta \cdot y)} s ds d\theta \quad (20)$$

$$\begin{aligned} &= \int_0^{\pi} \int_0^{\infty} G_{\theta}(s) e^{j2\pi s(x \cos \theta + y \sin \theta)} s ds d\theta \\ &\quad + \int_0^{\pi} \int_0^{\infty} G_{\theta+\pi}(s) e^{j2\pi s(x \cos(\theta+\pi) + y \sin(\theta+\pi))} s ds d\theta \end{aligned} \quad (21)$$

We note that $G_{\theta+\pi}(s) = G_{\theta}(-s)$, and let $\hat{s} = -s$.

Tomographic reconstruction

Continuing:

$$\begin{aligned}
 g(x, y) &= \int_0^\pi \int_0^\infty G_\theta(s) e^{j2\pi s(x \cos \theta + y \sin \theta)} s \, ds \, d\theta \\
 &\quad + \int_0^\pi \int_0^{-\infty} G_\theta(\hat{s}) e^{j2\pi \hat{s}(x \cos \theta + y \sin \theta)} (-\hat{s}) \, (-d\hat{s}) \, d\theta \quad (22)
 \end{aligned}$$

$$= \int_0^\pi \left[\int_{-\infty}^\infty |s| G_\theta(s) e^{j2\pi s(x \cos \theta + y \sin \theta)} ds \right] d\theta \quad (23)$$

$$= \int_0^\pi \left[\int_{-\infty}^\infty |s| G_\theta(s) e^{j2\pi s \hat{x}} ds \right] d\theta \quad (24)$$

Tomographic reconstruction

This means we have four steps:

- 1 From the captured projection signal $g_\theta(\hat{x})$ (i.e., Radon transform), compute

$$G_\theta(s) = \mathcal{F}\{g_\theta(\hat{x})\} \quad (25)$$

- 2 Multiply in the frequency domain (i.e., filter):

$$\tilde{G}_\theta(s) = |s|G_\theta(s) \quad (26)$$

- 3 Compute the inverse Fourier transform, and backprojection:

$$\tilde{g}_\theta(\hat{x}) = \mathcal{F}^{-1}\{\tilde{G}_\theta(s)\} \quad (27)$$

$$\tilde{g}(x, y) = \tilde{g}_\theta(x \cos \theta + y \sin \theta) \quad (28)$$

Tomographic reconstruction

- 4 Sum for all angle:

$$g(x, y) = \int_0^\pi \tilde{g}(x, y) d\theta \quad (29)$$

This is known as the *filtered backprojection (FBP)* method.

The “filtered” step (i.e., step 2) is what sets it apart from the naïve algorithm at the beginning of this lecture.