Useful functions and their transforms (II)

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ELEC8501: The Fourier Transform and Its Applications

(First Semester, 2023–24)

https://www.eee.hku.hk/~elec8501

1. Some useful 2D functions
2. Fourier transform of some useful 2D functions
3. The delta function in 2D
4. Fourier transform related to the delta function

2D functions

We now consider 2D function $f(x, y)$

time $\rightarrow$ space (spatial domain)

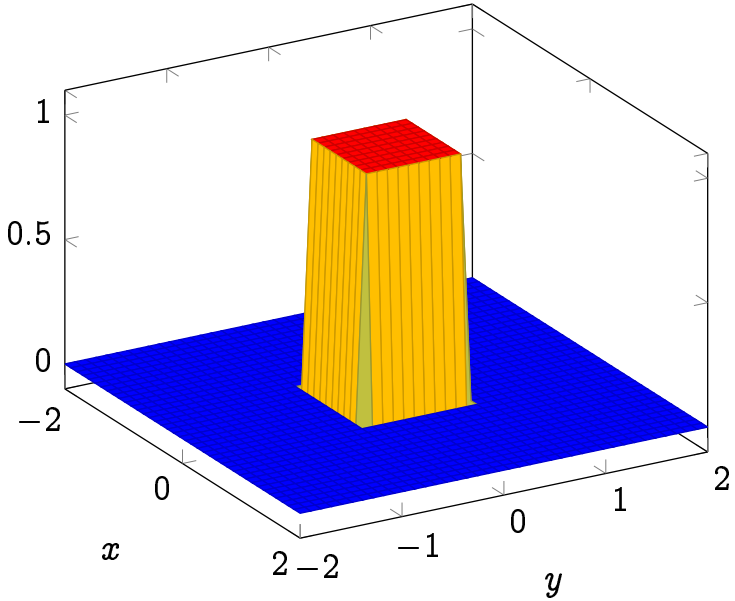
visualization is harder...

2D rect function

We define a 2D rect function as

$$\boxed{{}^2\Pi(x,y) = \begin{cases} 1 & |x| < \frac{1}{2} \text{ and } |y| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}} \tag{1}$$

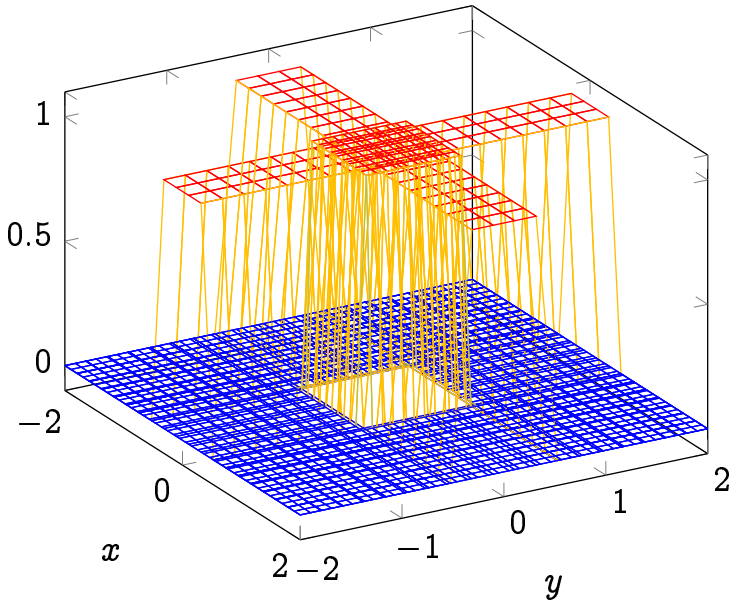
(The superscript "2" is often omitted when there's no ambiguity)



separable function

An important characteristic is that this is a *separable function* (in Cartesian coordinates), where we can express as $f(x, y) = f_X(x) f_Y(y)$.

$$\boxed{{}^2\Pi(x, y) = \Pi(x)\Pi(y)} \tag{2}$$



polar coordinates

In 2D, we can express functions not only in Cartesian coordinates, but also in polar coordinates.

Separable in Cartesian coordinates:

$$\boxed{f(x,y) = f_X(x) f_Y(y)} \tag{3}$$

Separable in polar coordinates:

$$\boxed{f(r,\theta) = f_R(r) f_\Theta(\theta)} \tag{4}$$

Furthermore, a function is *circularly symmetric* if:

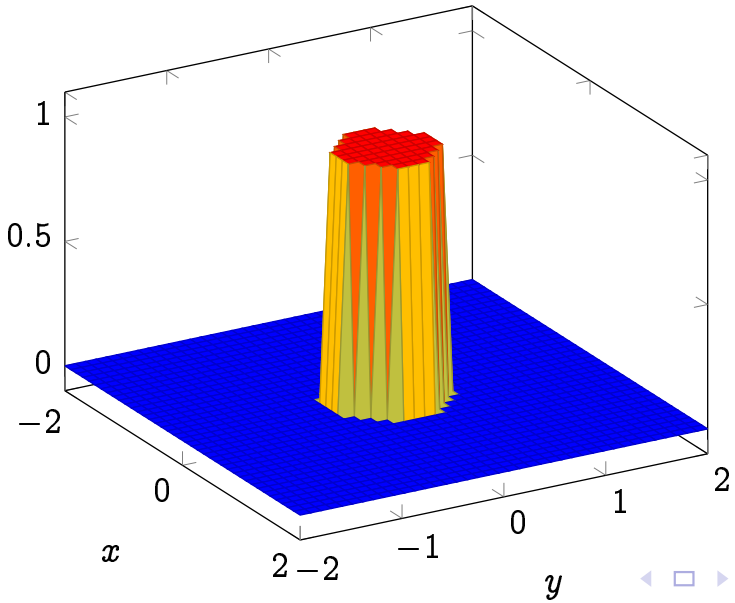
$$\boxed{f(r,\theta) = f_R(r)} \tag{5}$$

circ function

We define the *circ function* as

$$\Pi(r) = \begin{cases} 1 & |r| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Usually, it is the *context* that tells us it's a 2D function in polar coordinates, rather than a 1D function!

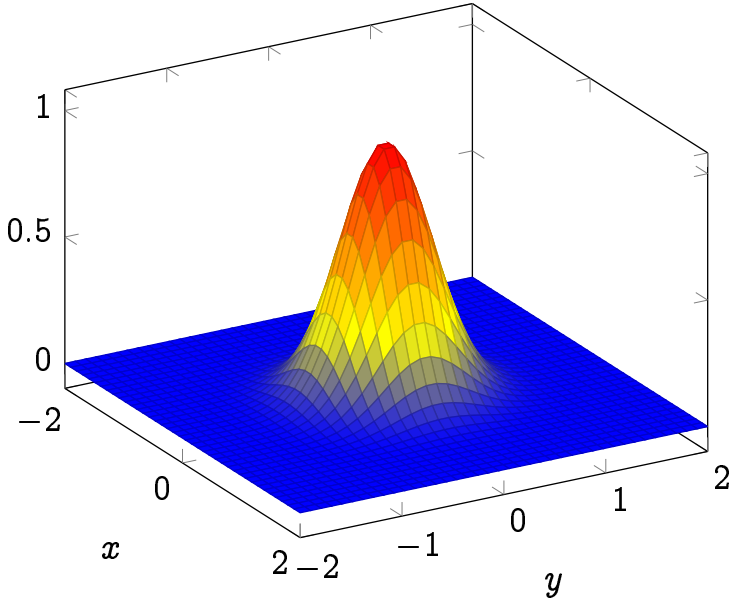


2D Gaussian

A special case: the 2D Gaussian function is separable in both Cartesian and polar coordinates, actually circularly symmetric

$$f(x,y) = e^{-\pi(x^2+y^2)} \tag{7}$$

$$f(r,\theta) = e^{-\pi r^2} \tag{8}$$



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2D Fourier transform

Fourier transform of a 2D function $f(x, y)$ is another 2D function $F(u, v)$

frequency $\rightarrow$ spatial frequency

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(xu+yv)}\,dx\,dy \tag{9}$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(xu+yv)}\,du\,dv \tag{10}$$

We can write ${}^{2}\mathcal{F}\{f(x,y)\} = F(u,v)$, but usually the superscript "2" is omitted when there's no ambiguity

n-D Fourier transform

If we write $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$, then $\mathbf{x} \cdot \mathbf{u} = xu + yv$, and therefore

$$F(\mathbf{u}) = \int_{-\infty}^{\infty} f(\mathbf{x}) e^{-j2\pi(\mathbf{x}\cdot\mathbf{u})} \, d\mathbf{x} \tag{11}$$

$$f(\mathbf{x}) = \int_{-\infty}^{\infty} F(\mathbf{u}) e^{j2\pi(\mathbf{u}\cdot\mathbf{x})} \, d\mathbf{u} \tag{12}$$

We can extend this to higher dimensions.

Separable functions

2D Fourier transform is much simpler if the function is separable, i.e., $f(x,y) = f_X(x) f_Y(y)$.

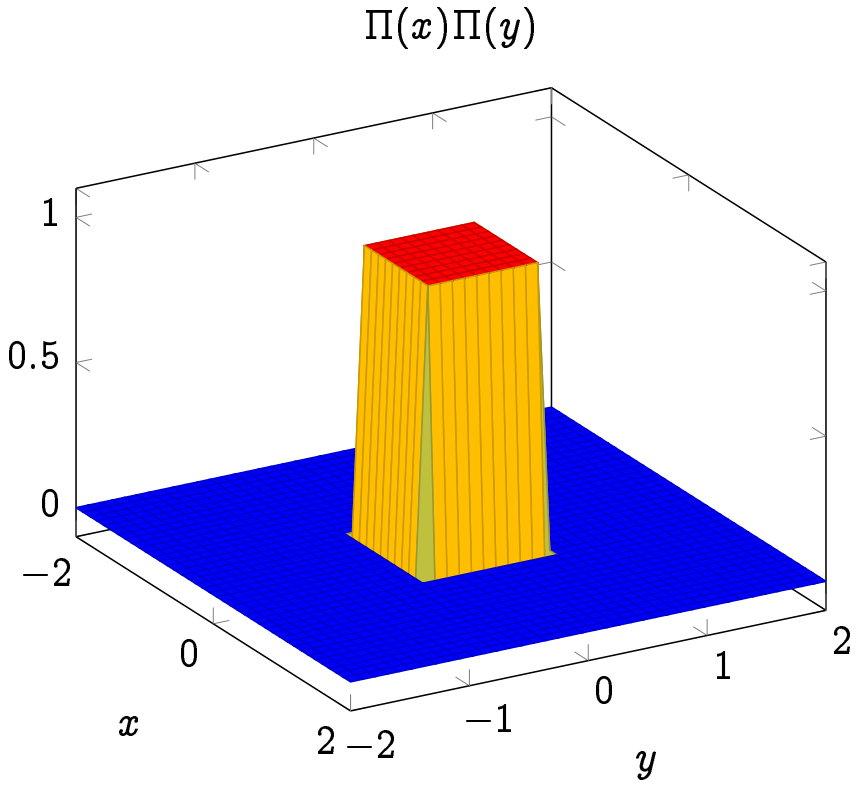
$$\mathcal{F}\{f(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(xu+yv)}\, dx\, dy \tag{13}$$

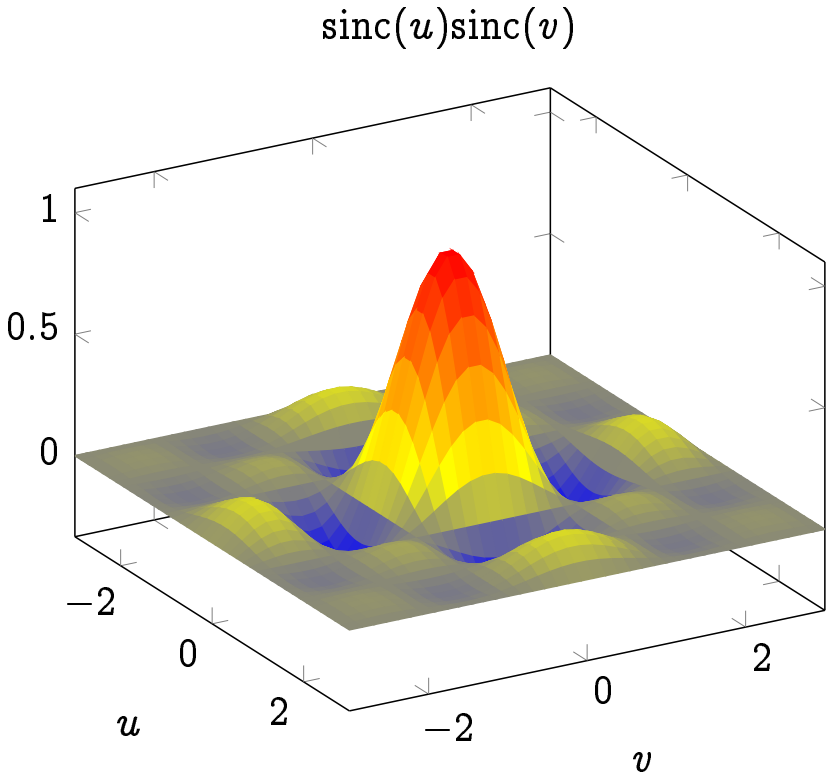
$$= \left(\int_{-\infty}^{\infty} f_X(x) e^{-j2\pi ux} dx\right)\left(\int_{-\infty}^{\infty} f_Y(y) e^{-j2\pi vy} dy\right) \tag{14}$$

$$\boxed{F(u,v) = F_U(u) F_V(v)} \tag{15}$$

(Note: This is NOT convolution!)

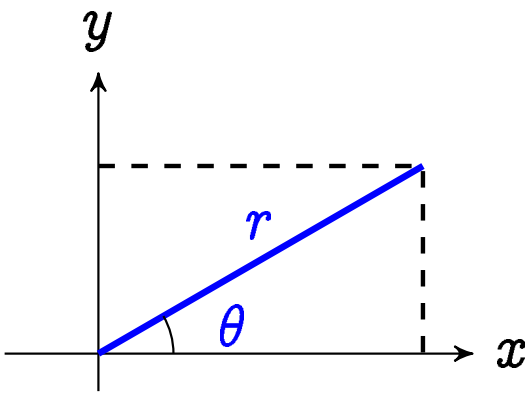
An example

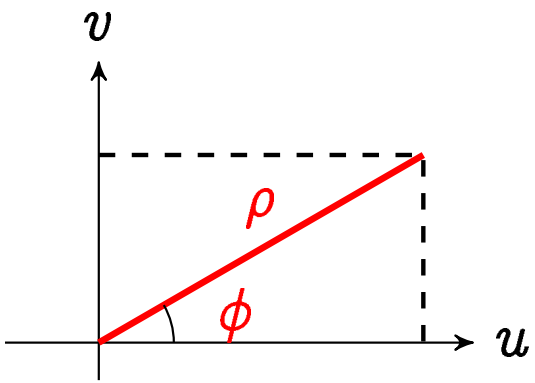

$\Pi(x)\Pi(y)$
1
0.5
0
−2
0
2
x
−2
−1
0
1
2
y


$\mathrm{sinc}(u)\mathrm{sinc}(v)$
1
0.5
0
−2
0
2
u
−2
0
2
v

separable in polar coordinates

We should do coordinate transformation when the function is separable in polar coordinates.





$$x = r\cos\theta \qquad u = \rho\cos\phi \tag{16}$$

$$y = r\sin\theta \qquad v = \rho\sin\phi \tag{17}$$

$$r = \sqrt{x^2 + y^2} \qquad \rho = \sqrt{u^2 + v^2} \tag{18}$$

$$\tan\theta = \frac{y}{x} \qquad \tan\phi = \frac{v}{u} \tag{19}$$

separable in polar coordinates

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(xu+yv)}\,dx\,dy \tag{20}$$

$$F(\rho,\phi) = \int_{0}^{2\pi}\int_{0}^{\infty} f_R(r)f_\Theta(\theta)e^{-j2\pi(r\cos\theta\cdot\rho\cos\phi+r\sin\theta\cdot\rho\sin\phi)}r\,dr\,d\theta \tag{21}$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} f_R(r)f_\Theta(\theta)e^{-j2\pi r\rho\cos(\theta-\phi)}r\,dr\,d\theta \tag{22}$$

For circularly symmetric functions: $f_\Theta(\theta)$ is a constant (e.g. $=1$)

$$\boxed{F_\rho(\rho) = \int_{0}^{\infty} r f_R(r)\left(\int_{0}^{2\pi} e^{-j2\pi r\rho\cos(\theta-\phi)}d\theta\right)dr} \tag{23}$$

Bessel functions

In dealing with circular objects, we often find the Bessel functions at work! (cf. sinusoidal for 1D)

$$J_0(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-jr\cos\theta} d\theta \tag{24}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(r\cos\theta) d\theta \tag{25}$$

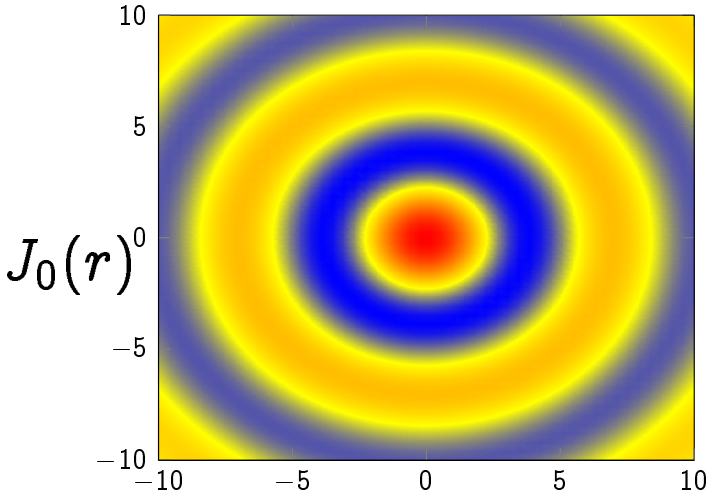
J means Bessel function of the "first kind"; 0 means "order zero".

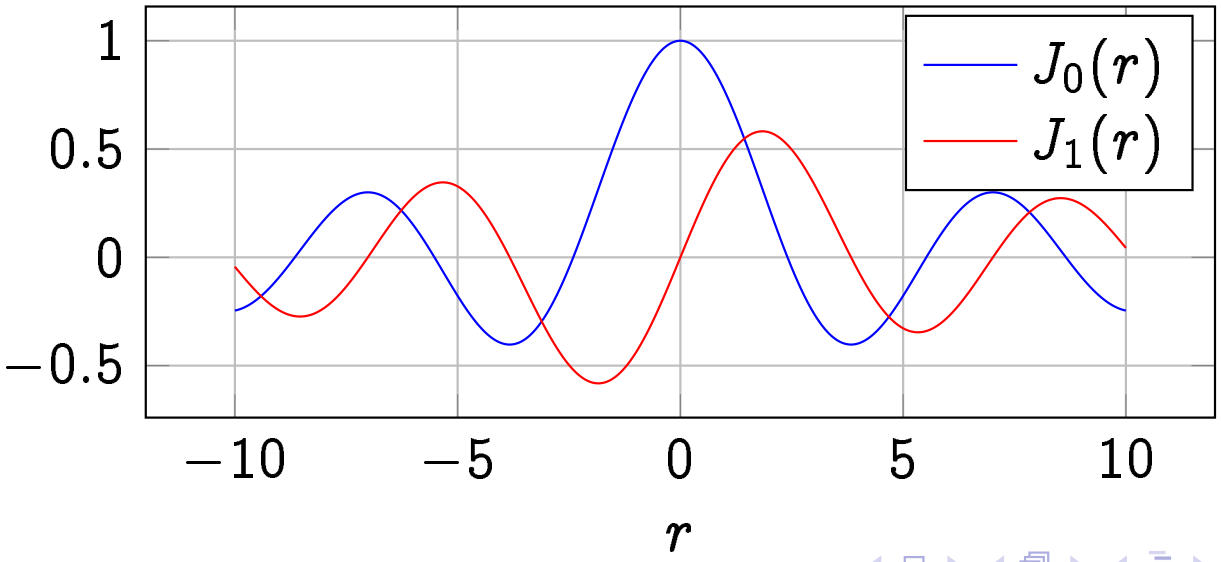
Also relevant for us is the Bessel function of the first kind, order one:

$$J_1(r) = \frac{1}{r}\int_0^{r} u J_0(u) du \tag{26}$$

Bessel functions

What they look like:





Fourier-Bessel transform

Putting the Bessel function in Eq. (23), and with similar derivations for the inverse Fourier transform, we have:

$$F_\rho(\rho) = 2\pi \int_0^\infty r f_R(r) J_0(2\pi r \rho)\, dr \tag{27}$$

$$f_R(r) = 2\pi \int_0^\infty \rho F_\rho(\rho) J_0(2\pi r \rho)\, d\rho \tag{28}$$

- 2D Fourier transform of a circularly symmetric function is also circularly symmetric
- This is also known as the Fourier-Bessel transform, or Hankel transform
- We may use $\mathcal{B}\{\cdot\}$ to denote the transform, e.g.,

$$\mathcal{B}\{f_R(r)\} = F_\rho(\rho) \tag{29}$$

Fourier-Bessel transform of circ

We want to compute the 2D Fourier transform of the circ function $\Pi(r)$:

$$\mathcal{B}\{\Pi(r)\} = 2\pi \int_0^{1/2} r(1) J_0(2\pi r \rho)\, dr \tag{30}$$

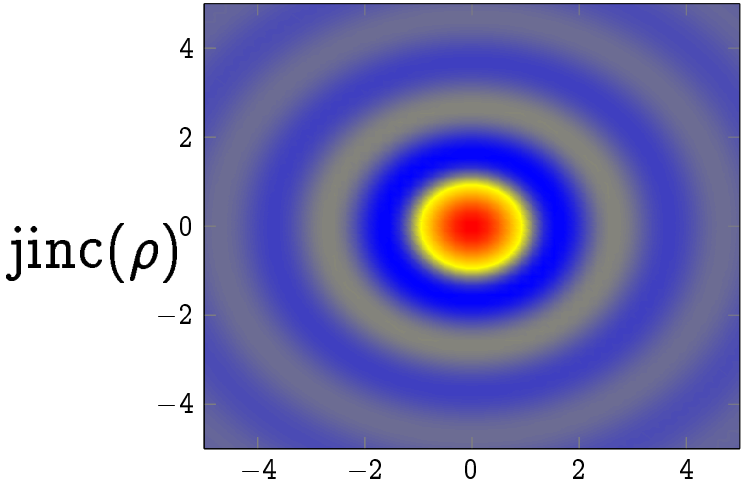
Let $\hat{r} = 2\pi r \rho$, so $r = \dfrac{\hat{r}}{2\pi\rho}$ and $dr = \dfrac{d\hat{r}}{2\pi\rho}$

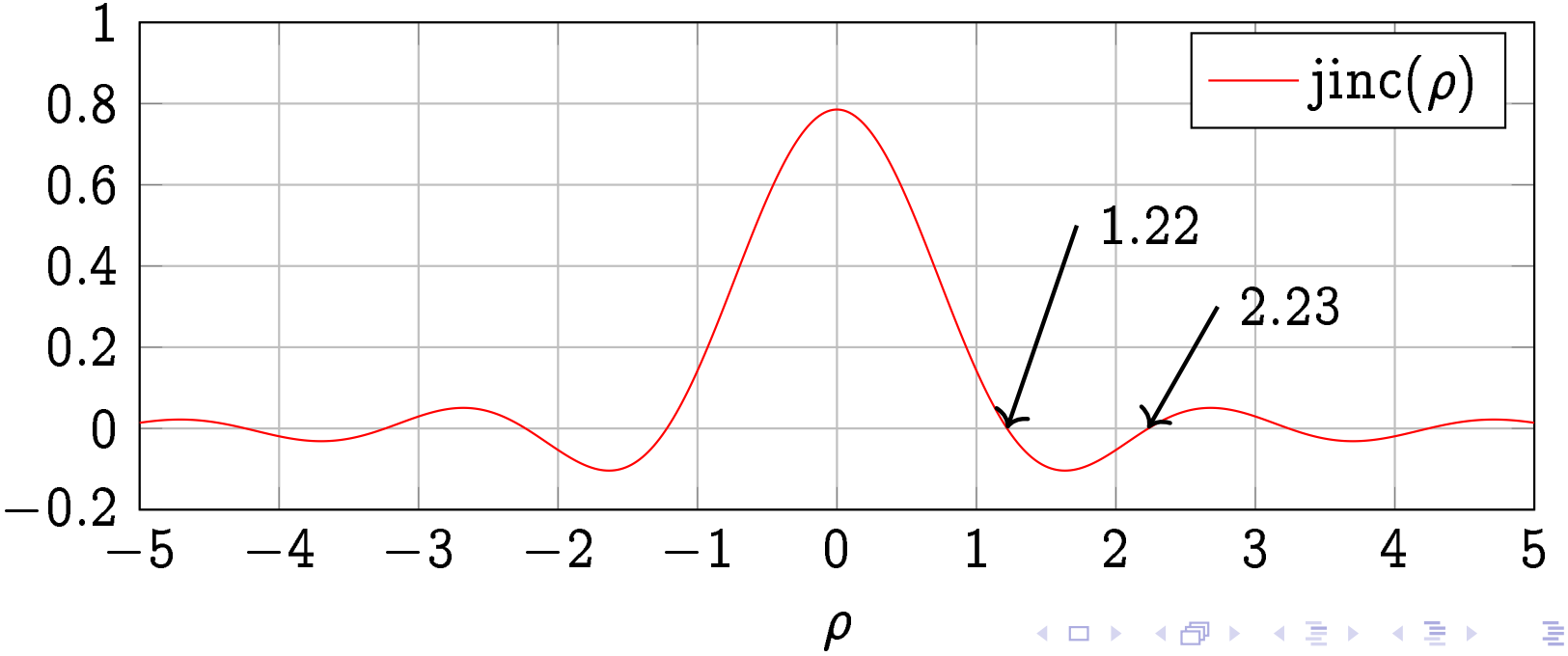
$$\mathcal{B}\{\Pi(r)\} = \frac{2\pi}{(2\pi\rho)^2} \underbrace{\int_0^{\pi\rho} \hat{r} J_0(\hat{r}) d\hat{r}}_{\pi\rho J_1(\pi\rho)} \tag{31}$$

$$= \frac{J_1(\pi\rho)}{2\rho} \triangleq \mathrm{jinc}(\rho) \tag{32}$$

$$\boxed{\Pi(r) \supset \mathrm{jinc}(\rho)} \tag{33}$$

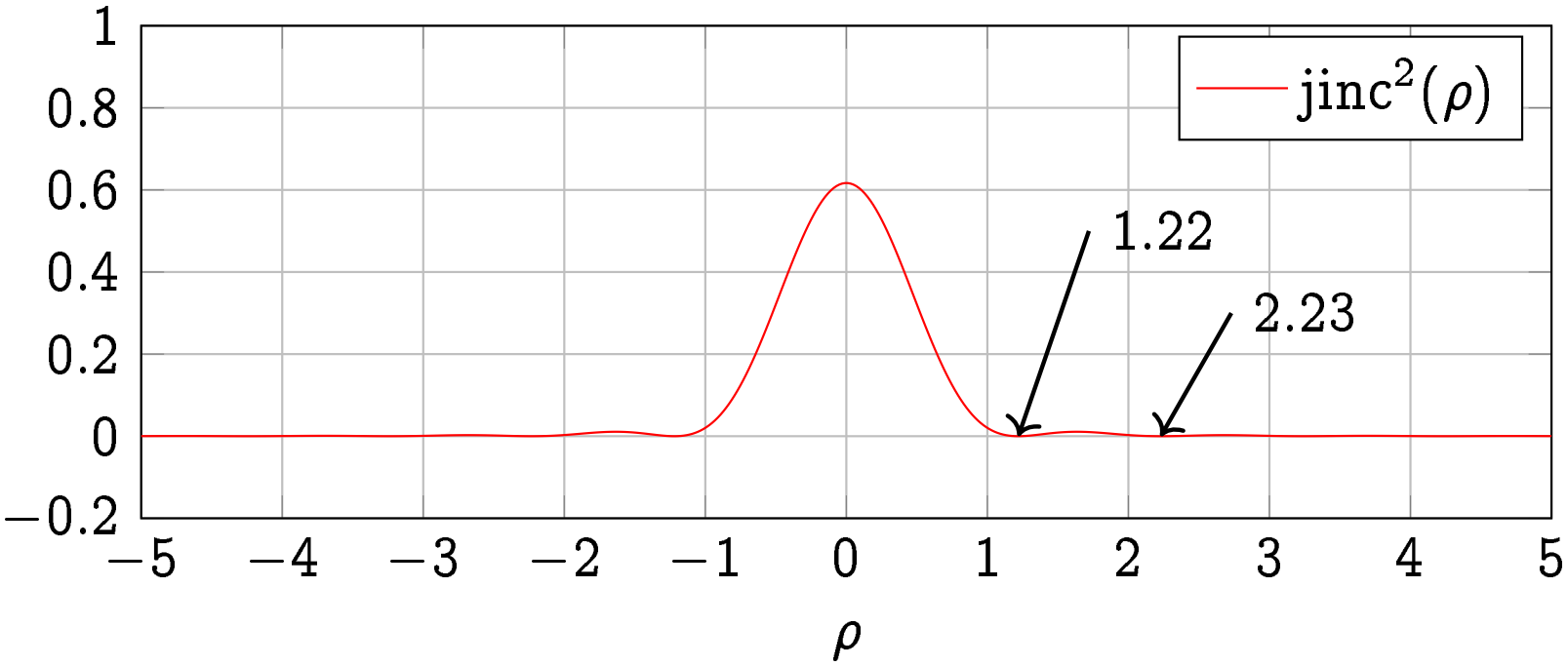
Fourier-Bessel transform of circ





use in imaging

The value 1.22 is very important when considering the resolution of an imaging system. We will also encounter $\text{jinc}^2(\rho)$, which is known as the Airy pattern. The inverse Fourier-Bessel transform is called the Chinese hat function, $\text{chat}(r)$.



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2D delta function

We can extend the concept of delta function to 2D, e.g.,

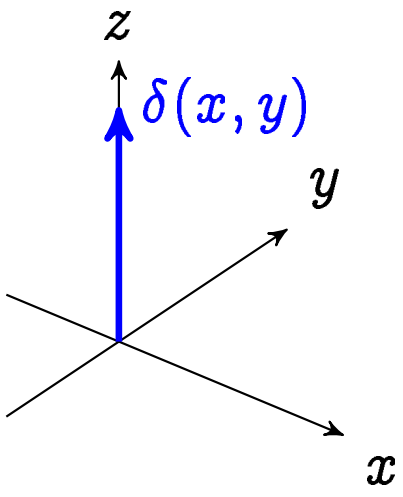
$$\boxed{{}^2\delta(x,y) \triangleq \lim_{\tau\to 0} \frac{1}{\tau^2}\, {}^2\Pi\left(\frac{x}{\tau}, \frac{y}{\tau}\right)} \tag{34}$$

As usual, we often ignore the "2" and overload the symbols.

This delta function has *unit volume*.

2D delta function

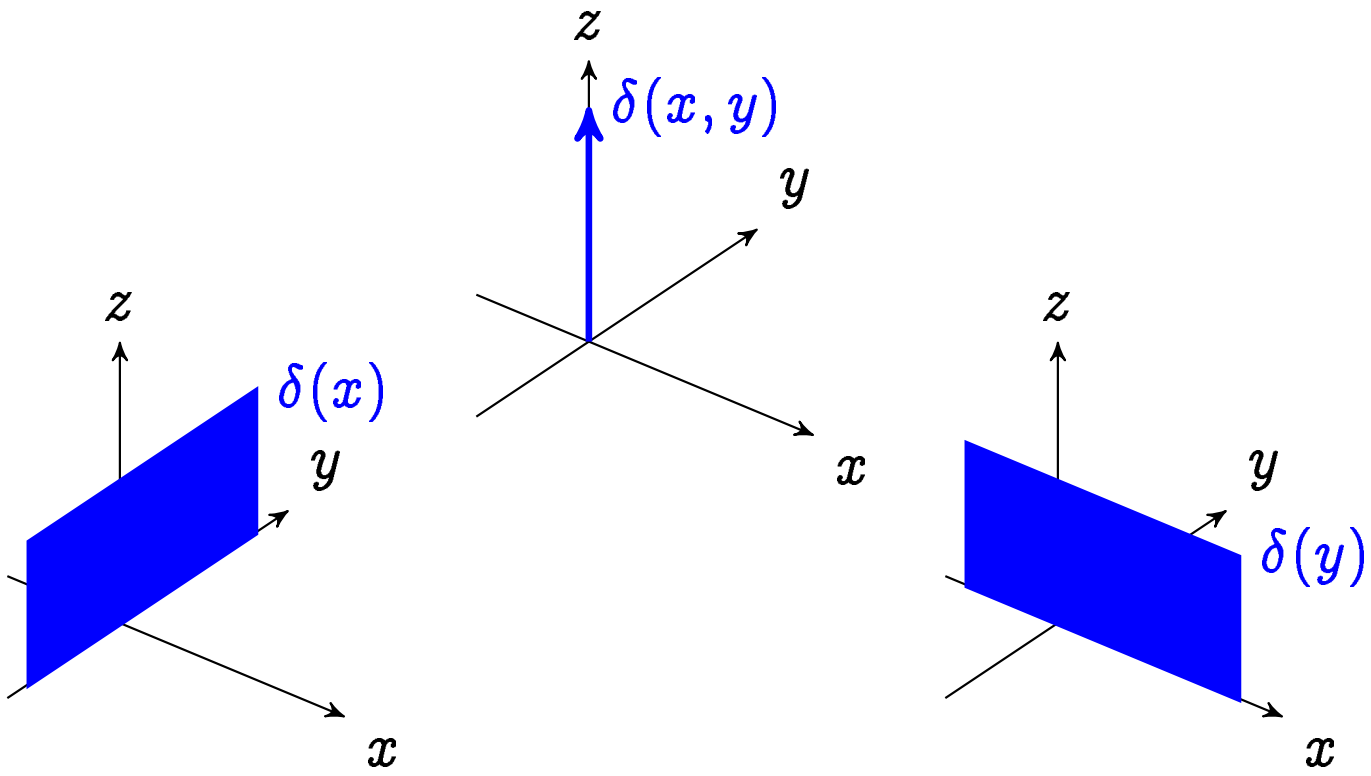
We can also have $\delta(x)$ and $\delta(y)$ in 2D.



$$\boxed{\delta(x, y) = \delta(x)\delta(y)} \tag{35}$$

2D delta function

We can also have $\delta(x)$ and $\delta(y)$ in 2D.



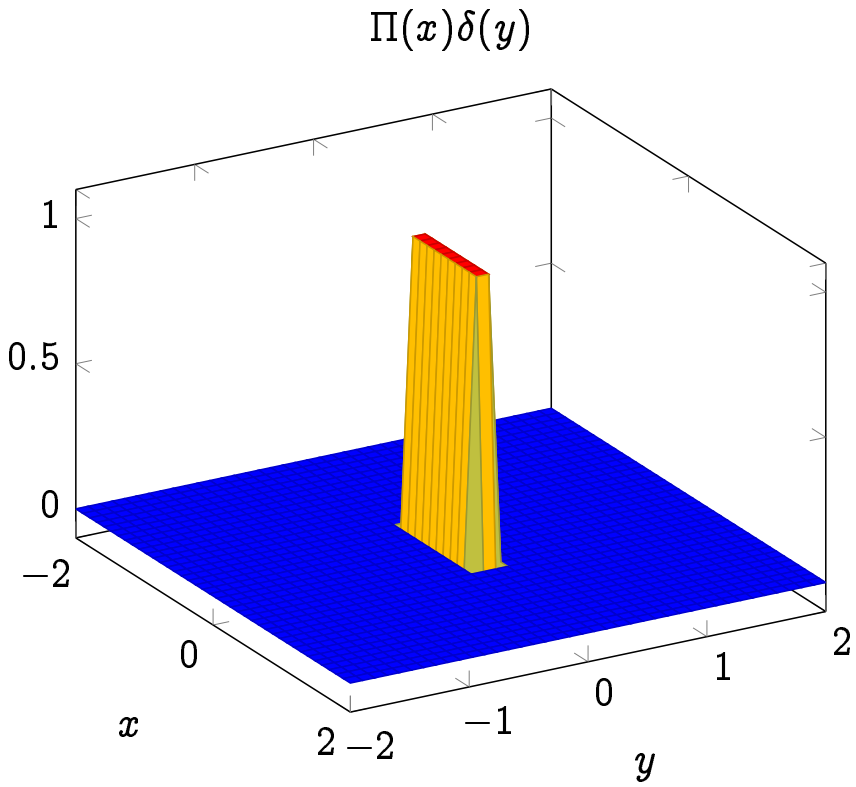
$$\boxed{\delta(x,y) = \delta(x)\delta(y)} \tag{35}$$

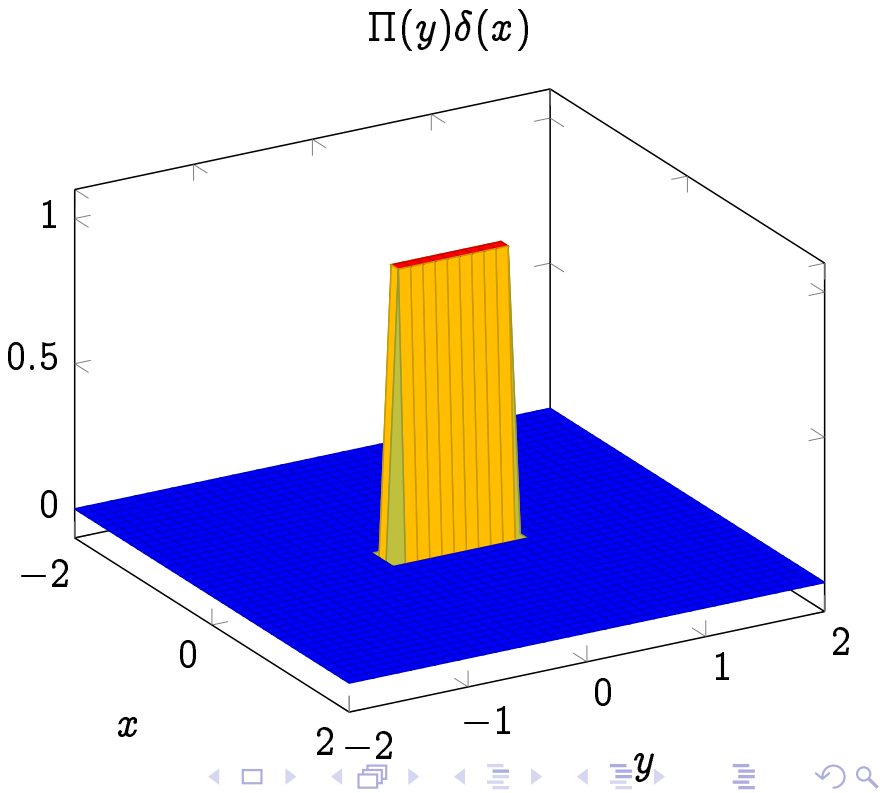
2D rect function

Another view of the 2D rect function:

$$\boxed{{}^{2}\Pi(x,y) = [\Pi(x)\delta(y)] ** [\Pi(y)\delta(x)]} \tag{36}$$

(we often use * instead of ** to denote 2D convolution)





delta in polar coordinates

We can also have delta functions in polar coordinates.

Does $\delta(x, y) = \delta(r)$?

delta in polar coordinates

We can also have delta functions in polar coordinates.

Does $\delta(x, y) = \delta(r)$?

$$\boxed{\delta(x, y) = \frac{1}{\pi |r|} \delta(r)} \tag{37}$$

(Reason: The proportional constant arises with the change of coordinates.)

Try to picture what $\delta(r - k)$ is, for a constant k.

line delta

Back to Cartesian coordinates, is

$$\delta(y - mx - c) \tag{38}$$

a "line delta" at the line $y = mx + c$?

line delta

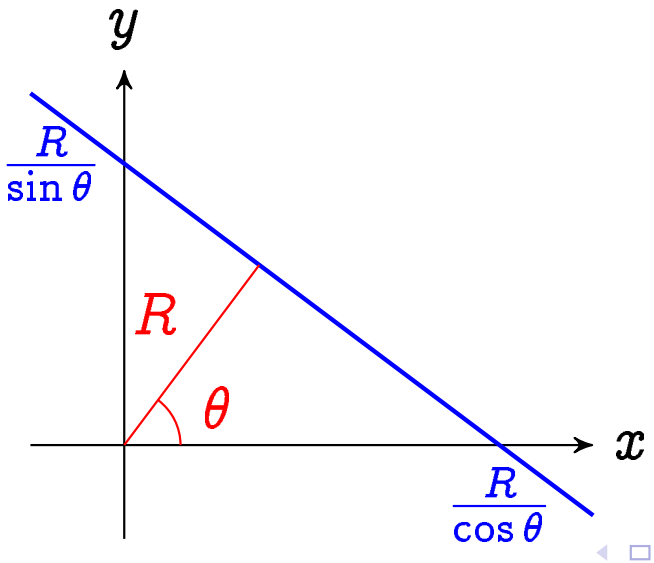
Back to Cartesian coordinates, is

$$\delta(y - mx - c) \tag{38}$$

a "line delta" at the line $y = mx + c$?

ANS: Need to be careful on the "strength" of the delta... the proper representation is

$$\delta(R - x\cos\theta - y\sin\theta) \tag{39}$$



shah in 2D

We can also have a shah function in 2D:

$$\boxed{\sqcup\!\sqcup(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x-n, y-m)} \tag{40}$$

Try to picture:

- $\sqcup\!\sqcup(x,y)$ ("bed of nails")
- $\sqcup\!\sqcup(x)$ in 2D ("grating")
- $\sqcup\!\sqcup(x)\delta(y)$ ("row of spikes")

shah in 2D

How do we represent a "bed of nails" with a distance (X, Y) apart in the two directions? Remember $\delta(ax) = \frac{1}{|a|}\delta(x)$

shah in 2D

How do we represent a "bed of nails" with a distance (X, Y) apart in the two directions? Remember $\delta(ax) = \frac{1}{|a|}\delta(x)$

$$\delta(x - nX) = X\delta\left(\frac{x}{X} - n\right) \tag{41}$$

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(x-n, y-m) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} XY\delta\left(\frac{x}{X} - n, \frac{y}{Y} - m\right) \tag{42}$$

$$= XY \sqcup\!\sqcup \left(\frac{x}{X} - n, \frac{y}{Y} - m\right) \tag{43}$$

Therefore, the unit strength deltas would be

$$\boxed{\frac{1}{XY} \sqcup\!\sqcup \left(\frac{x}{X}, \frac{y}{Y}\right)} \tag{44}$$

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delta

We can extend from 1D to 2D. What is the Fourier transform of $\delta(x, y)$?

delta

We can extend from 1D to 2D. What is the Fourier transform of $\delta(x, y)$?

$$\boxed{\delta(x, y) \supset 1} \tag{45}$$

delta

What is the Fourier transform of $\delta(x)$?

delta

What is the Fourier transform of $\delta(x)$?

$$\boxed{\delta(x) \supset \delta(v)} \tag{46}$$

(Hint: $\delta(x) = \delta(x) \cdot 1$)

cosine and sine

What is the Fourier transform of $\cos \pi(x+y)$?

cosine and sine

What is the Fourier transform of $\cos \pi(x+y)$?

$$\cos \pi(x+y) = \cos \pi x \cos \pi y - \sin \pi x \sin \pi y \tag{47}$$

$$\cos \pi x \cos \pi y = \left(\frac{1}{2}\left[\delta(u-\tfrac{1}{2}) + \delta(u+\tfrac{1}{2})\right]\right)\left(\frac{1}{2}\left[\delta(v-\tfrac{1}{2}) + \delta(v+\tfrac{1}{2})\right]\right) \tag{48}$$

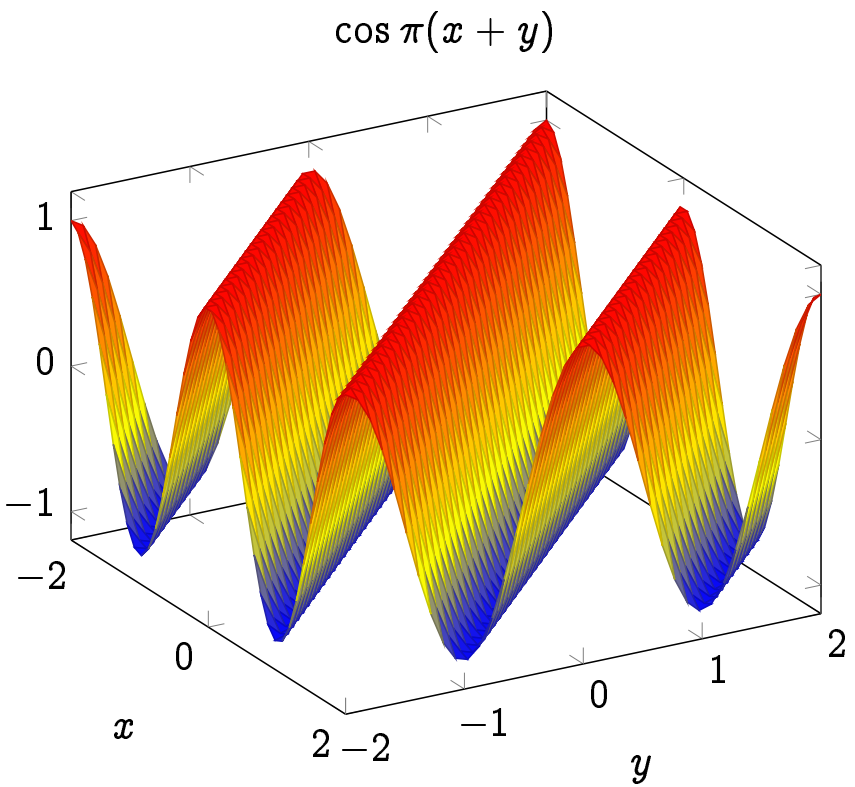
$$= \frac{1}{4}\left[\delta(u^-, v^-) + \delta(u^-, v^+) + \delta(u^+, v^-) + \delta(u^+, v^+)\right] \tag{49}$$

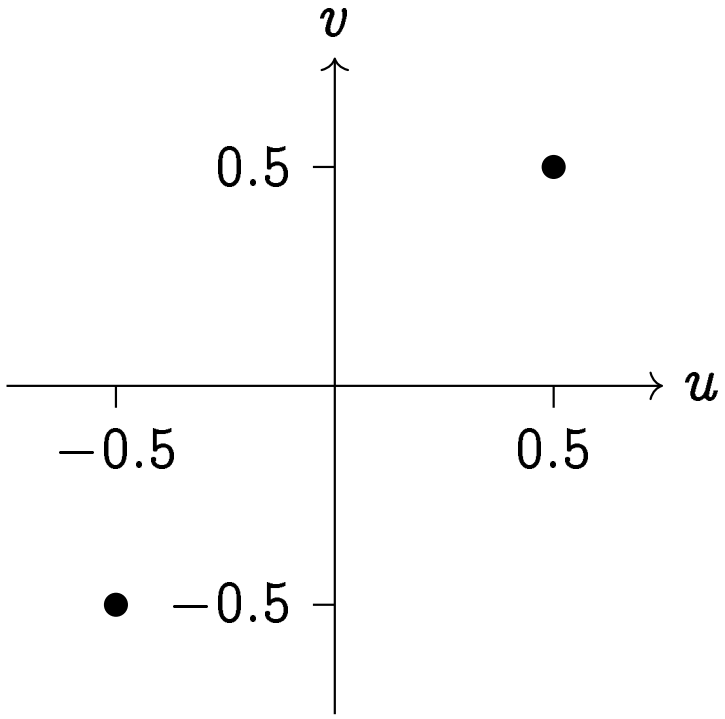
$$-\sin \pi x \sin \pi y = -\left(\frac{j}{2}\left[\delta(u+\tfrac{1}{2}) - \delta(u-\tfrac{1}{2})\right]\right)\left(\frac{j}{2}\left[\delta(v+\tfrac{1}{2}) - \delta(v-\tfrac{1}{2})\right]\right) \tag{50}$$

$$= \frac{1}{4}\left[\delta(u^-, v^-) - \delta(u^-, v^+) - \delta(u^+, v^-) + \delta(u^+, v^+)\right] \tag{51}$$

cosine and sine

$$\cos \pi(x+y) \supset \frac{1}{2}\left[\delta(u-\tfrac{1}{2}, v-\tfrac{1}{2}) + \delta(u+\tfrac{1}{2}, v+\tfrac{1}{2})\right] \tag{52}$$





shah function

What is $\mathcal{F}\{\sqcup\!\sqcup(x, y)\}$?

shah function

What is $\mathcal{F}\{\text{Ш}(x,y)\}$?

$$\text{Ш}(x,y) \supset \text{Ш}(u,v) \tag{53}$$

$$\text{Ш}(x) \supset \text{Ш}(u)\delta(v) \tag{54}$$

$$\text{Ш}(x)\delta(y) \supset \text{Ш}(u) \tag{55}$$