Useful functions and their transforms (I)

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ELEC8501: The Fourier Transform and Its Applications

(First Semester, 2023–24)

https://www.eee.hku.hk/~elec8501

1 Some useful functions

2 Fourier transform of some useful functions

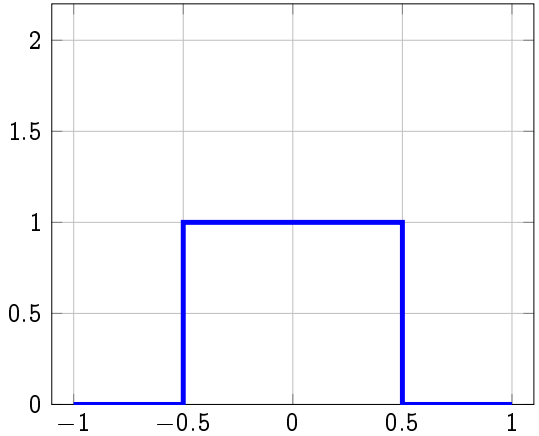
3 The delta function

4 Fourier transform related to the delta function

(1) rect function

In Fourier transform, we often deal with piecewise linear functions. The simplest one is a *rect function*.

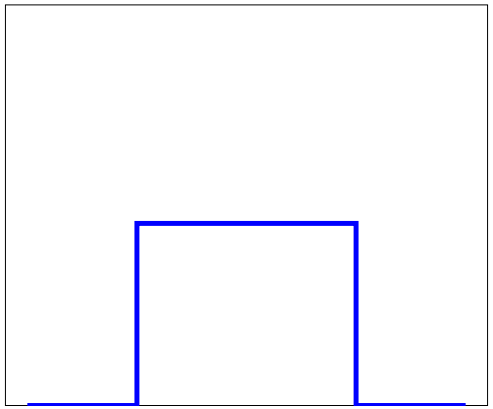
$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$



rect function

The rect function has a unit area. But we can *scale*, *shift*, and *stretch*, with

$$\boxed{f(t) = a\Pi\left(\frac{t-b}{c}\right)} \tag{2}$$



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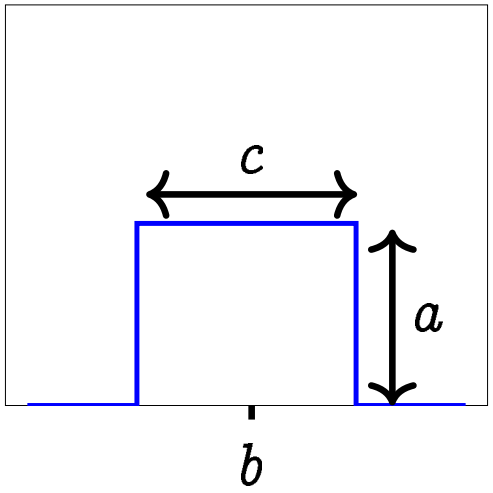


rect function

The rect function can be used to:

1. change integration limit:

$$\boxed{\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)dt = \int_{-\infty}^{\infty} f(t)\Pi(t)dt} \tag{3}$$

2. extract a segment of a function:

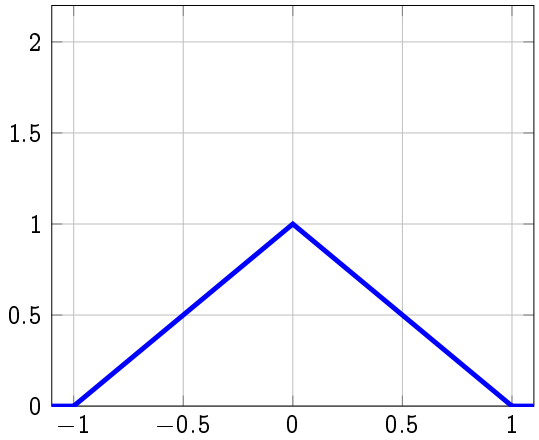
$$\boxed{f(t)\Pi\left(\frac{t-b}{c}\right)} \tag{4}$$

3. approximate a function as piecewise linear (Riemann integral)
4. interpolate a discrete signal (more later)

(2) triangle function

Next we have the *triangle function* (of unit area).

$$\Lambda(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$



Sketch $f(t) = a\Lambda\left(\frac{t-b}{c}\right)$.

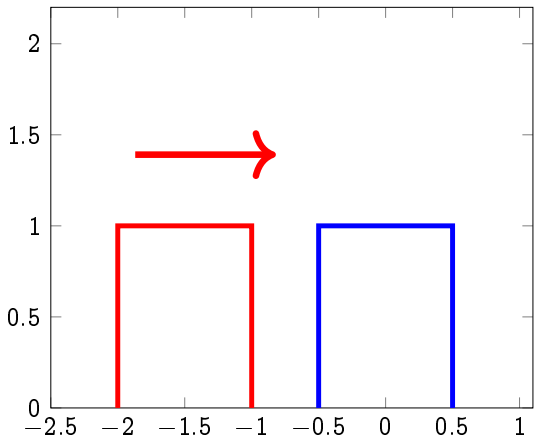
triangle function

A very important relationship:

$$\boxed{\Pi(t) * \Pi(t) = \Lambda(t)} \tag{6}$$

We use $*$ to represent convolution:

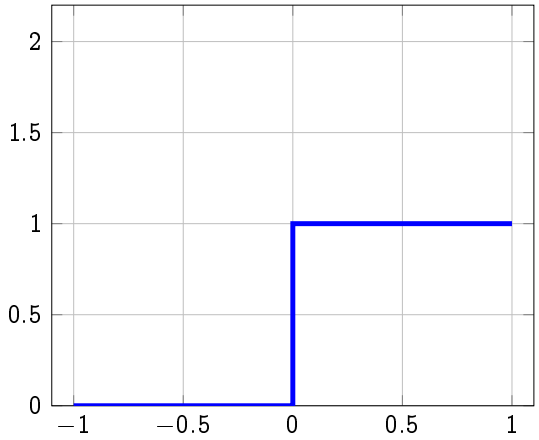
$$\boxed{f(t) * g(t) \triangleq \int_{-\infty}^{\infty} f(u)g(t-u)du} \tag{7}$$



(3) step function

Next, we have the Heaviside step function

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$



Sketch $f(t) = aH(t-b)$.

step function

1. Make a signal causal (kill the signal before $t = 0$)

$$\boxed{g(t) = H(t)f(t)} \tag{9}$$

2. Integrate a function:

$$H(t) * f(t) = \int_{-\infty}^{\infty} f(u) \underbrace{H(t-u)}_{=1 \text{ when } t-u>0} du \tag{10}$$

$$= \int_{-\infty}^{t} f(u)du \tag{11}$$

Differentiating both sides gives:

$$\boxed{\frac{d}{dt}[H(t) * f(t)] = f(t)} \tag{12}$$

step function

1. How do you represent $\int_0^t f(u)du$?

step function

1. How do you represent $\int_0^t f(u)du$?

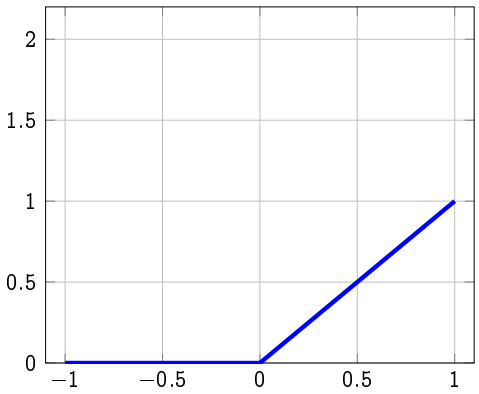
$$\boxed{H(t) * [H(t)f(t)] = \int_0^t f(u)du} \tag{13}$$

2. We can create a rect from a step:

$$\boxed{\Pi(t) = H(t + \tfrac{1}{2}) - H(t - \tfrac{1}{2})} \tag{14}$$

step function

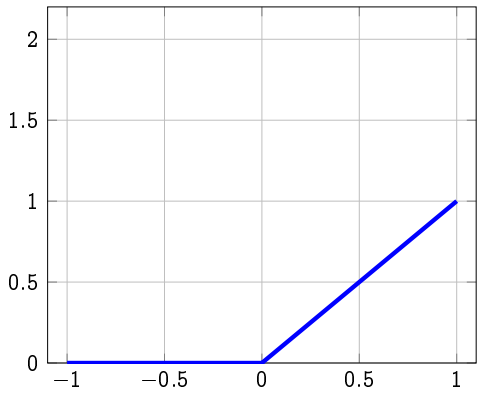
We may want to have a ramp function:



We are accustomed to calling it the ReLU: $f(t) = t^+ = \max(0, t)$

step function

We may want to have a ramp function:



We are accustomed to calling it the ReLU: $f(t) = t^+ = \max(0, t)$

1. Can use a step function to create a ramp:

$$\boxed{f(t) = tH(t)} \tag{15}$$

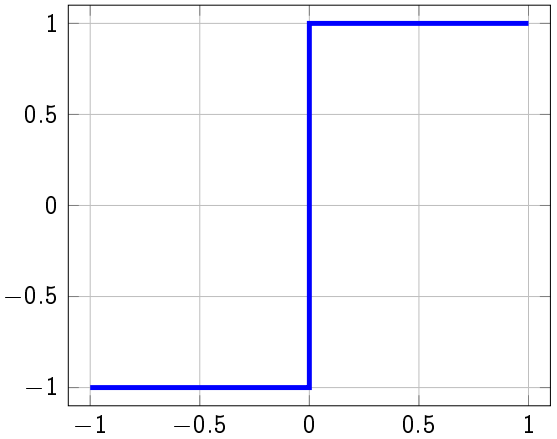
We also have $f(t) = \int_{-\infty}^{t} H(u)du$. Therefore,

$$\boxed{f(t) = H(t) * H(t)} \tag{16}$$

(4) signum function

We define the "sign" function (but pronounced as "signum") as

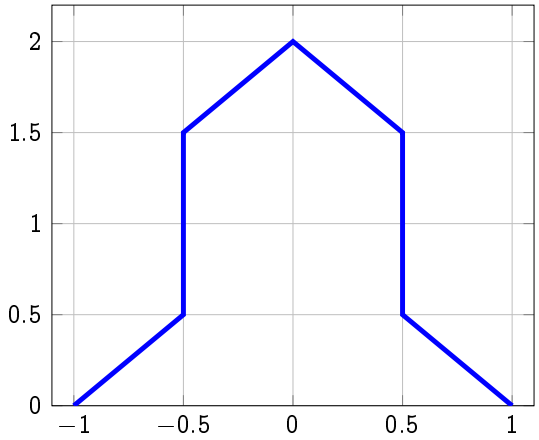
$$\boxed{\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & \text{otherwise.} \end{cases}} \tag{17}$$



$$\boxed{\operatorname{sgn}(t) = 2H(t) - 1} \tag{18}$$

Some exercises

1. Express algebraically



2. Sketch $H(t)e^{-t}$
3. Sketch and show $\frac{d}{dt}\Lambda(t) = -\Pi\left(\frac{t}{2}\right)\operatorname{sgn}(t)$
4. Prove that $\Pi(t^2) = \Pi(t/\sqrt{2})$. Hint: How to interpret

$$\Pi(f(t)) = \begin{cases} 1 & |f(t)| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

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(1) rect function

Computer $\mathcal{F}\{\Pi(t)\}$ by direct integration (fill in the steps):

(1) rect function

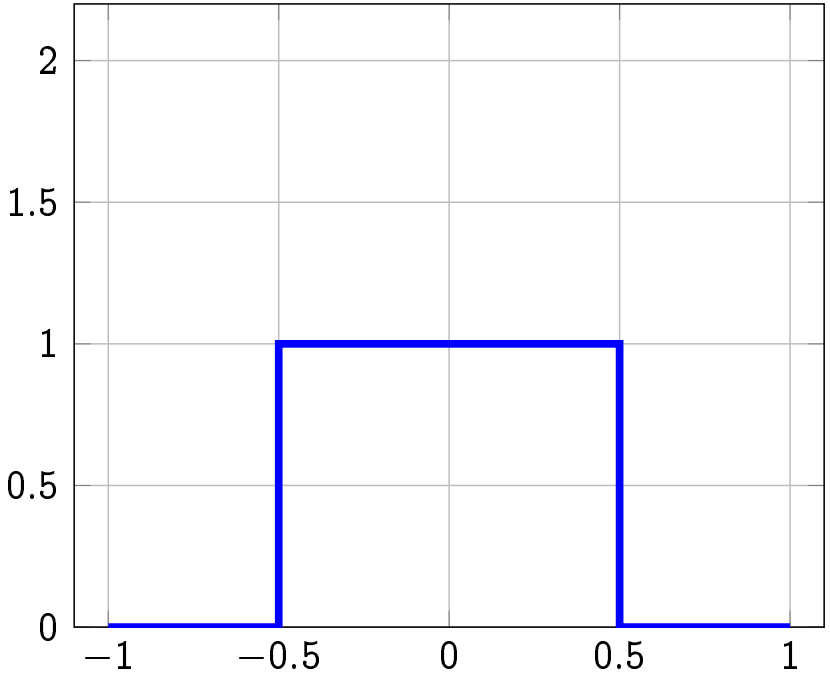
Computer $\mathcal{F}\{\Pi(t)\}$ by direct integration (fill in the steps):

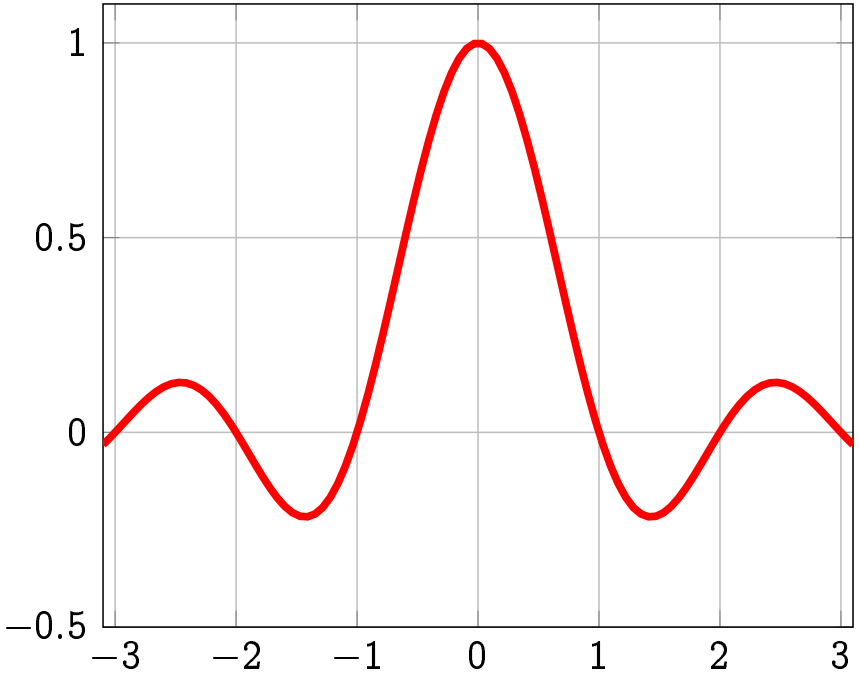
$$\boxed{\Pi(t) \supset \operatorname{sinc}(s)} \tag{20}$$

where $\operatorname{sinc}(s) \triangleq \dfrac{\sin \pi s}{\pi s}$

rect and sinc

Try to label the graphs:





(2) triangle function

Computer $\mathcal{F}\{\Lambda(t)\}$ by direct integration (try it yourself):

(2) triangle function

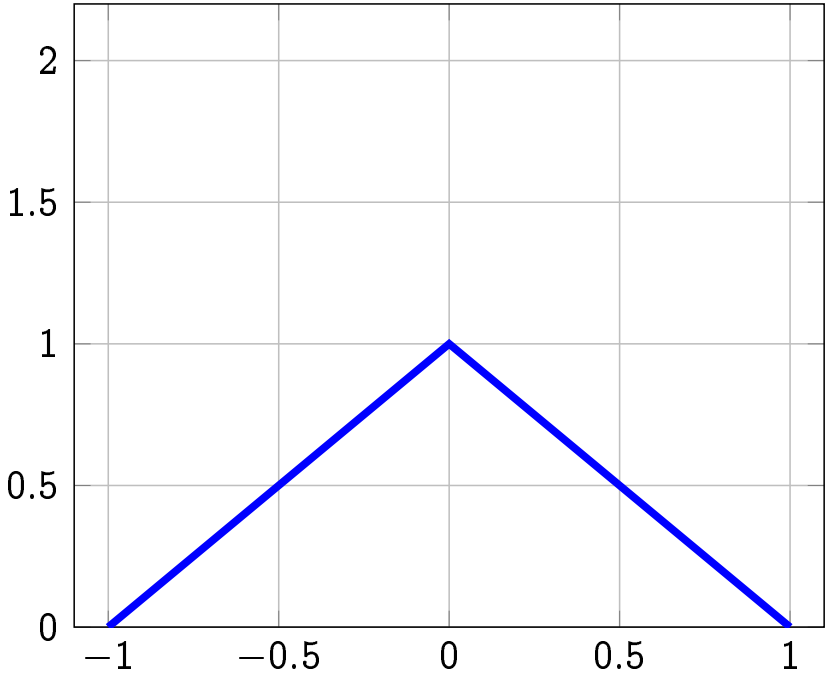
Computer $\mathcal{F}\{\Lambda(t)\}$ by direct integration (try it yourself):

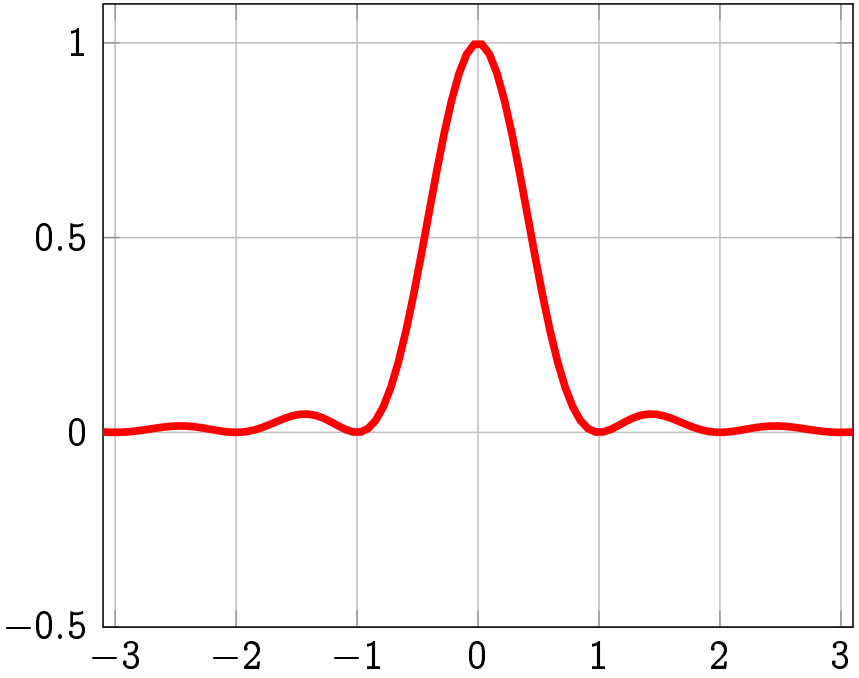
$$\boxed{\Lambda(t) \supset \operatorname{sinc}^2(s)} \tag{21}$$

(there is a better way to do this than direct integration)

triangle and sinc squared

Try to label the graphs:





(3) step function

$$\boxed{H(t) \supset \frac{1}{2}\delta(s) - \frac{j}{2\pi s}} \tag{22}$$

We will defer discussion of $\delta(s)$ in the meantime.

It's closely related to the signum function, so let's look into that first.

(4) signum function

Computer $\mathcal{F}\{\text{sgn}(t)\}$ by direct integration (try it yourself):

(4) signum function

Computer $\mathcal{F}\{\text{sgn}(t)\}$ by direct integration (try it yourself):

How do we deal with:

$$\mathcal{F}\{\text{sgn}(t)\} = j\left[\frac{\cos(2\pi st)}{2\pi s}\right]_{t=\infty} - \frac{j}{\pi s} \tag{23}$$

It fails a sufficient condition for the existence of Fourier transform: $\int_{-\infty}^{\infty} |f(t)| dt < \infty$

Fourier transform in the limit

A 3-step approach called "Fourier transform in the limit":

1. Define $f_N(t)$ such that $\lim_{N\to\infty} f_N(t) = f(t)$
2. Compute $F_N(s) = \mathcal{F}\{f_N(t)\}$
3. Define $F(s) \triangleq \lim_{N\to\infty} F_N(s)$

An example:

$$f_N(t) = \begin{cases} e^{-t/N} & t \geq 0 \\ -e^{t/N} & \text{otherwise.} \end{cases} \tag{24}$$

(Sketch it yourself.)

Fourier transform in the limit

1. Check $\lim_{N\to\infty} f_N(t) = \text{sgn}(t)$
2. Compute $F_N(s) = \mathcal{F}\{f_N(t)\}$ to obtain

$$F_N(s) = \frac{1}{\frac{1}{N} + j2\pi s} - \frac{1}{\frac{1}{N} - j2\pi s} \tag{25}$$

3. Therefore,

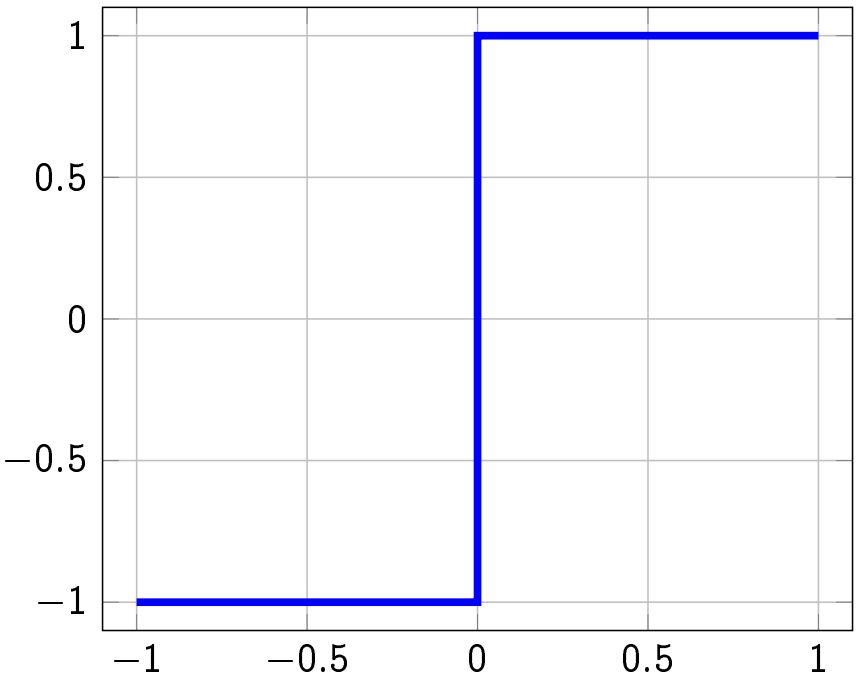
$$F(s) = \lim_{N\to\infty}\left(\frac{1}{\frac{1}{N} + j2\pi s} - \frac{1}{\frac{1}{N} - j2\pi s}\right) \tag{26}$$

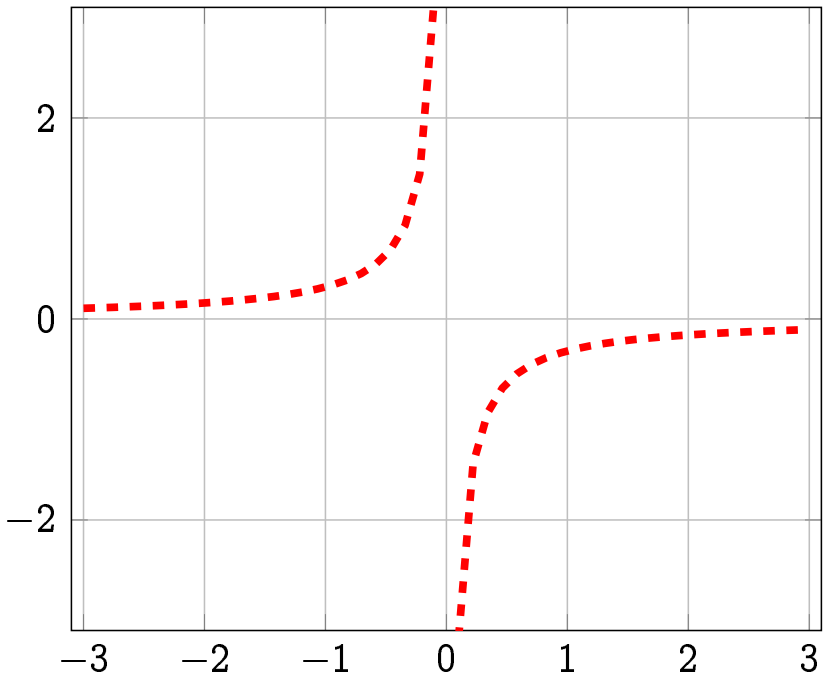
$$= \frac{1}{j2\pi s} + \frac{1}{j2\pi s} \tag{27}$$

$$= -\frac{j}{\pi s} \tag{28}$$

signum function

$$\boxed{\mathrm{sgn}(t) \supset -\frac{j}{\pi s}} \tag{29}$$





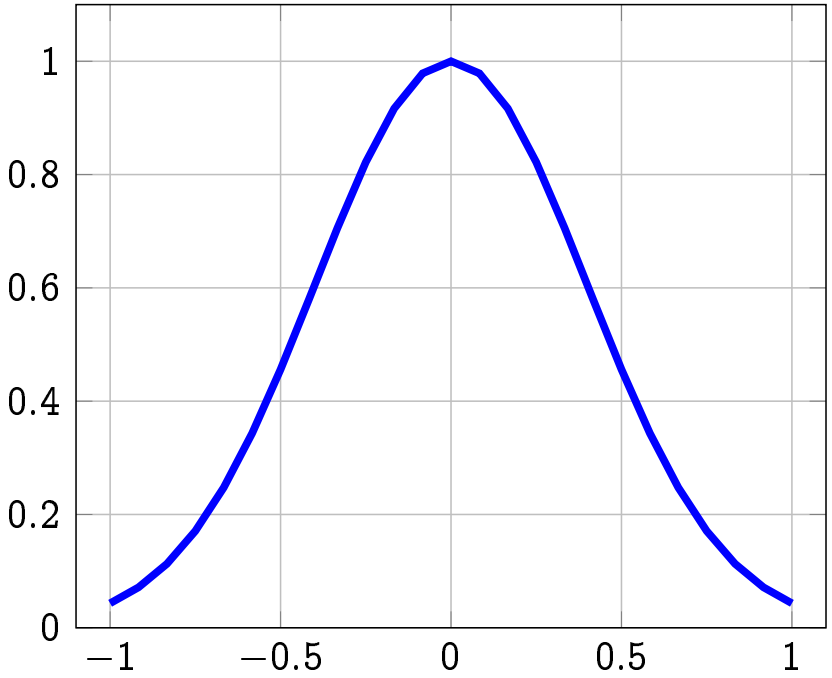
(5) Gaussian function

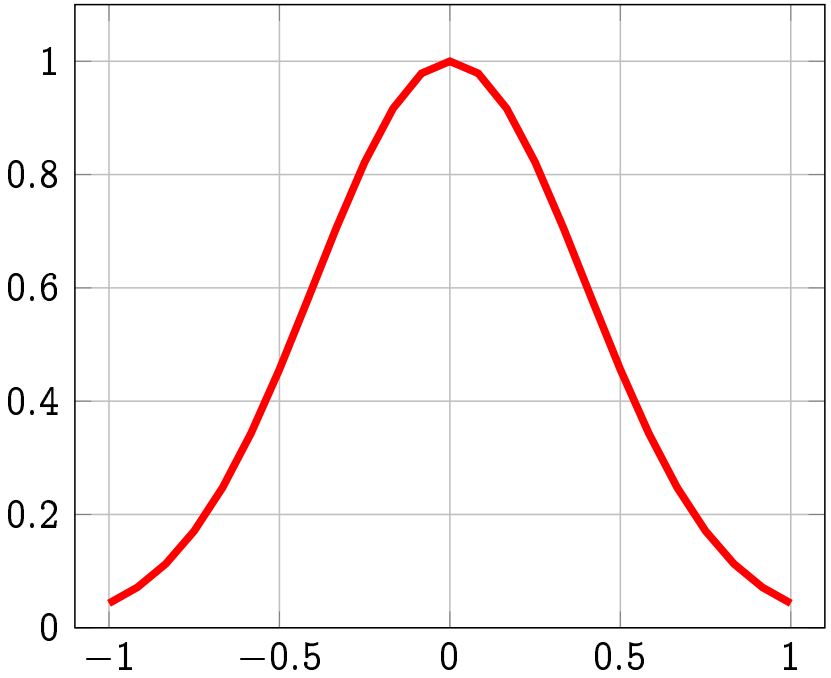
Computer $\mathcal{F}\left\{e^{-\pi t^2}\right\}$ by direct integration (try it yourself):

(5) Gaussian function

Computer $\mathcal{F}\left\{e^{-\pi t^2}\right\}$ by direct integration (try it yourself):

$$\boxed{e^{-\pi t^2} \supset e^{-\pi s^2}} \tag{30}$$





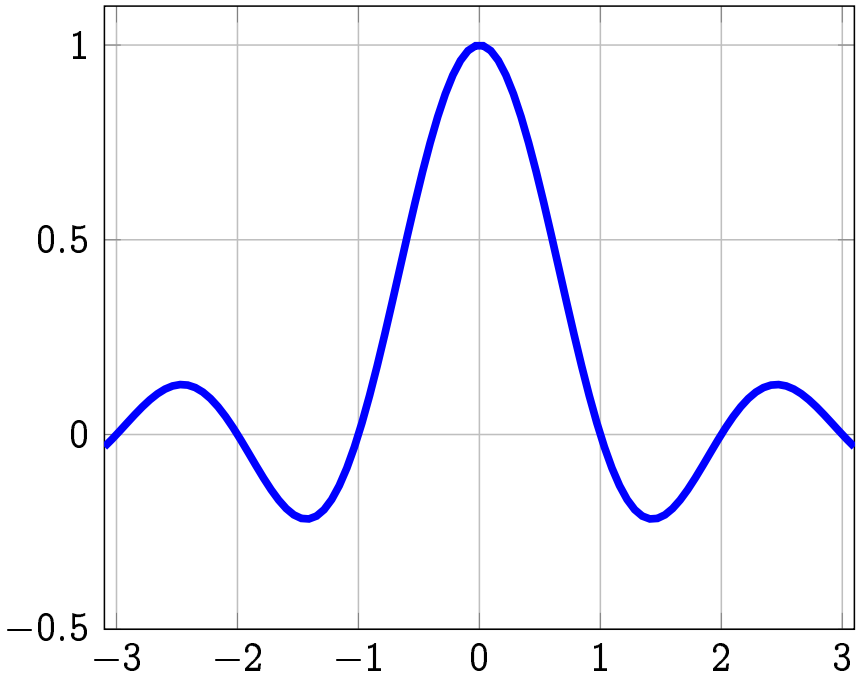
(6) sinc function

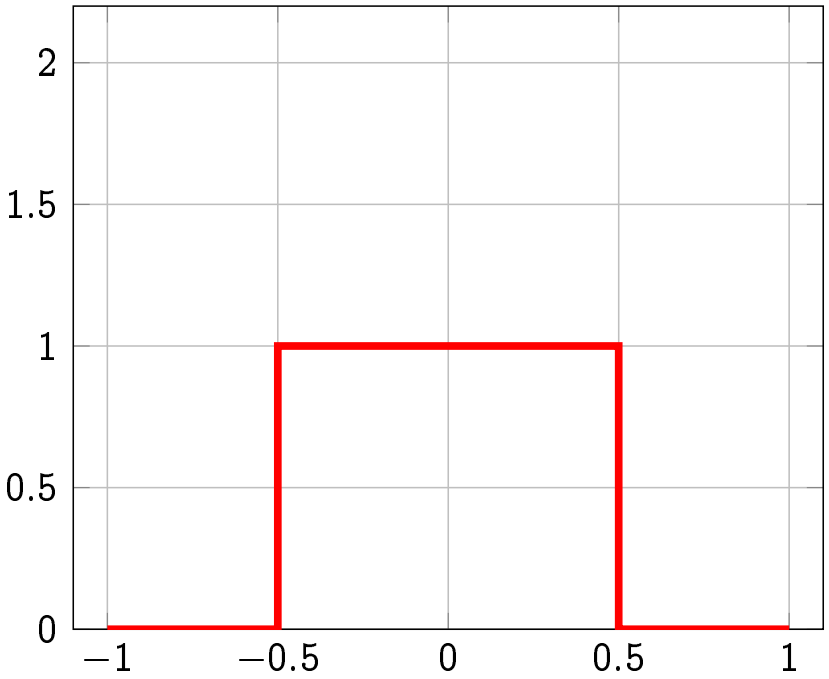
Computer $\mathcal{F}\{\text{sinc}(t)\}$ by direct integration (try it yourself):

(6) sinc function

Computer $\mathcal{F}\{\text{sinc}(t)\}$ by direct integration (try it yourself):

$$\boxed{\text{sinc}(t) \supset \Pi(s)} \tag{31}$$





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an impulse

To understand the behavior of a system, we often want to hit it with a short pulse (input) and observe what happens (output)

This short pulse is an "impulse"

What we hope to have: a function where

$$\begin{cases} f(t) = 0 \quad t \neq 0 \\ \int_{-\infty}^{\infty} f(t)dt = 1 \end{cases} \tag{32}$$

but it is not possible to have such a function!

delta function

So, we define the (Dirac) delta function as a *generalized function*.

$$\delta(t) \triangleq \lim_{\tau \to 0} \frac{1}{\tau} \Pi\left(\frac{t}{\tau}\right) \tag{33}$$

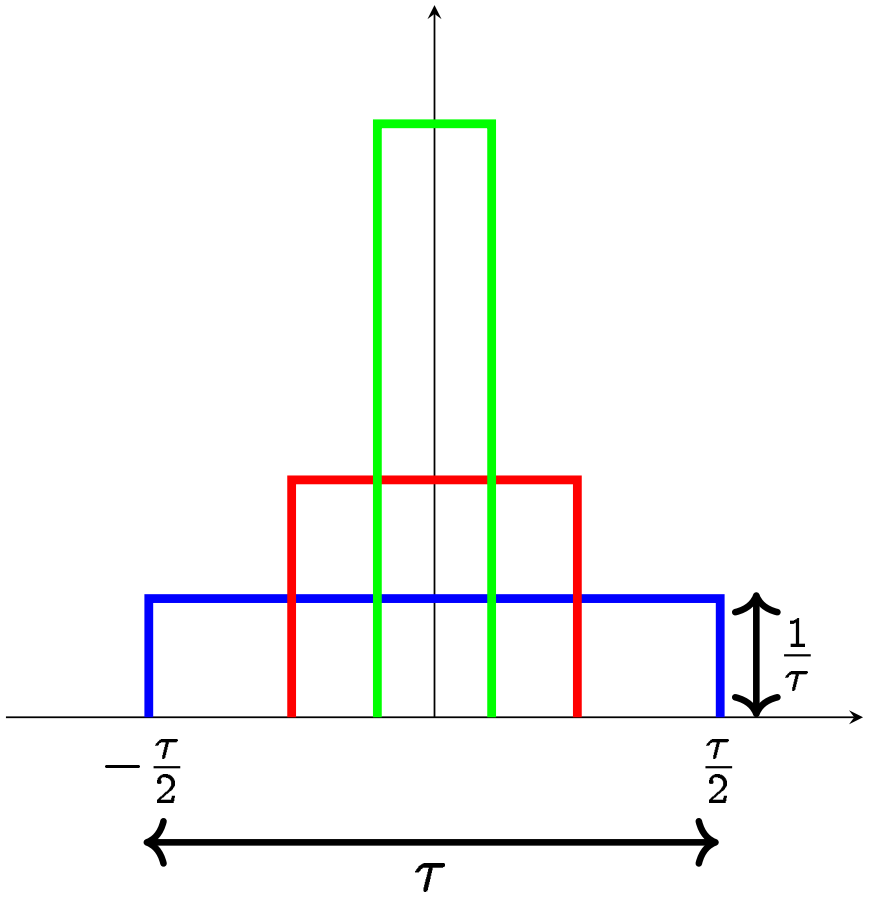
$$\delta(t) \triangleq \lim_{\tau \to 0} \frac{1}{\tau} \Lambda\left(\frac{t}{\tau}\right) \tag{34}$$

$$\delta(t) \triangleq \lim_{\tau \to 0} \frac{1}{\tau} e^{-\pi t^2/\tau^2} \tag{35}$$

$$\delta(t) \triangleq \lim_{\tau \to 0} \frac{1}{\tau} \mathrm{sinc}\left(\frac{t}{\tau}\right) \tag{36}$$

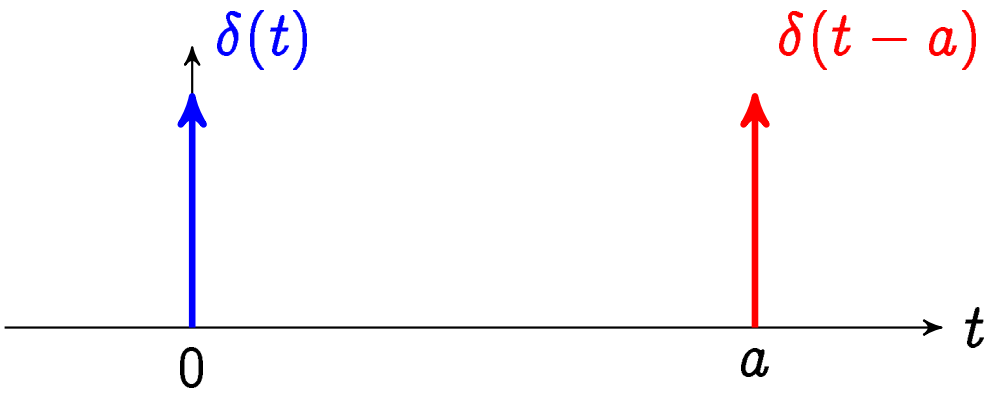
delta function

The delta function has unit area, and is zero everywhere except at the origin. For example, $\delta(t) \triangleq \lim_{\tau \to 0} \frac{1}{\tau} \Pi\left(\frac{t}{\tau}\right)$:



delta function properties

We draw $\delta(t)$ with an arrow:



The defining characteristic is the *sifting property*:

$$\boxed{\begin{aligned}\int_{-\infty}^{\infty} \delta(t) f(t) dt &= \int_{-\infty}^{\infty} \left[\lim_{\tau \to 0} \frac{1}{\tau} \Pi\left(\frac{t}{\tau}\right)\right] f(t) dt \\ &= f(0)\end{aligned}} \tag{37}$$

delta function properties

From the sifting properties, we can derive:

$$\boxed{\int_{-\infty}^{\infty} \delta(t-a) f(t) dt = f(a)} \tag{38}$$

$$\boxed{\delta(t) * f(t) = \int_{-\infty}^{\infty} f(u)\delta(t-u) du = f(t)} \tag{39}$$

$$\boxed{\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi st} dt = 1} \tag{40}$$

delta function properties

It is important to keep in mind that while $\delta(t)$ does not have a "height" (at $t = 0$), it has an "area", and therefore can be scaled.

Example: Express $\delta(at)$ for $a > 0$.

delta function properties

It is important to keep in mind that while $\delta(t)$ does not have a "height" (at $t = 0$), it has an "area", and therefore can be scaled.

Example: Express $\delta(at)$ for $a > 0$.

$$\begin{aligned}\delta(at) &\triangleq \lim_{\tau \to 0} \frac{1}{\tau} \Pi\left(\frac{at}{\tau}\right) \\ &= \frac{1}{a} \lim_{\tau \to 0} \frac{a}{\tau} \Pi\left(\frac{at}{\tau}\right) \\ &= \frac{1}{a} \lim_{\tau' \to 0} \frac{1}{\tau'} \Pi\left(\frac{t}{\tau'}\right)\end{aligned}$$

where $\tau' = \tau/a$. Generalizing,

$$\boxed{\delta(at) = \frac{1}{|a|}\delta(t)} \tag{41}$$

impulse and step

We can relate the delta function and the step function as follows. Since

$$\int_{-\infty}^{t} \delta(u)du = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \tag{42}$$

ignoring what happens at $t = 0$, we can say

$$\boxed{\int_{-\infty}^{t} \delta(u)du = H(t)} \tag{43}$$

Furthermore,

$$\boxed{\delta(t) = \frac{d}{dt}H(t)} \tag{44}$$

Note that by calculus, $\frac{d}{dt}2H(t) = 2\delta(t)$.

delta function properties

Some further relationships:

- $\delta(-t) = \delta(t)$
- $f(t)\delta(t) = f(0)\delta(t)$
- $t\delta(t) = 0$
- $\mathcal{F}^{-1}\{1\} = \delta(t) = \int_{-\infty}^{\infty} e^{j2\pi st} ds$

derivative of delta

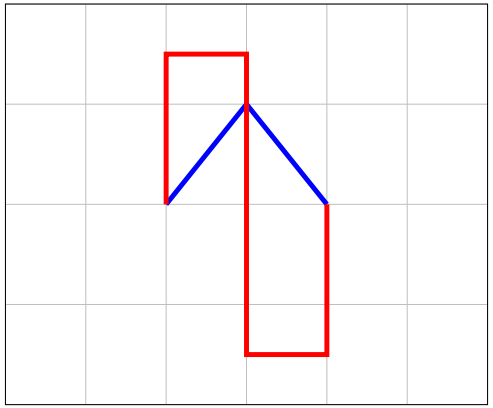
There are occasions (such as in the chain rule) that we need to calculate

$$\delta'(t) = \frac{d}{dt}\delta(t) \tag{45}$$

We can define it with the derivative-sifting property

$$\boxed{\delta'(t) * f(t) = \int_{-\infty}^{\infty} \delta'(t-u) f(u) du = f'(t)} \tag{46}$$

$$\frac{1}{\tau}\Lambda(\frac{t}{\tau}) \rightarrow \frac{1}{\tau^2}\Pi(\frac{t}{2\tau})\mathrm{sgn}(t)$$



delta derivative properties

Some further relationships:

- $\delta'(-t) = -\delta'(t)$
- $t\delta'(t) = -\delta(t)$
- $f(t)\delta'(t) = f(0)\delta'(t) - f'(0)\delta(t)$. Proof:

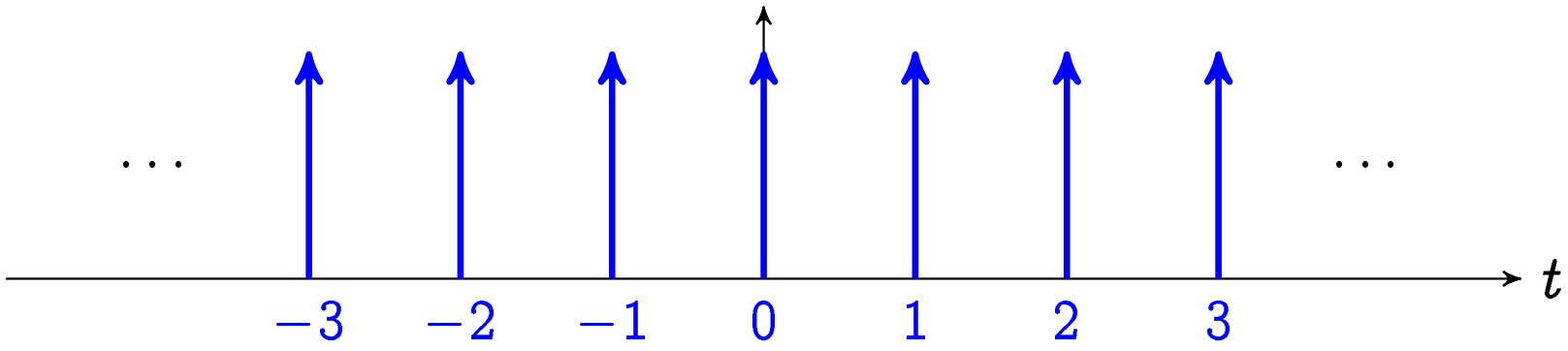
$$\begin{aligned} \frac{d}{dt} f(t)\delta(t) &= f'(t)\delta(t) + f(t)\delta'(t) \\ \frac{d}{dt} f(0)\delta(t) &= f'(0)\delta(t) + f(t)\delta'(t) \\ f(0)\delta'(t) - f'(0)\delta(t) &= f(t)\delta'(t) \end{aligned}$$

- $\int_{-\infty}^{\infty} t\delta'(t)dt = -1$
- $\delta^{(n)}(t) * f(t) = f^{(n)}(t)$

shah function

Define the shah (comb) function as an infinite train of delta functions:

$$\boxed{\text{Ш}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)} \tag{47}$$



shah function

The shah function is most useful in *sampling* and *replication*:

$$\text{Ш}(t)f(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t-n) \tag{48}$$

$$\text{Ш}(t) * f(t) = \sum_{n=-\infty}^{\infty} f(t-n) \tag{49}$$

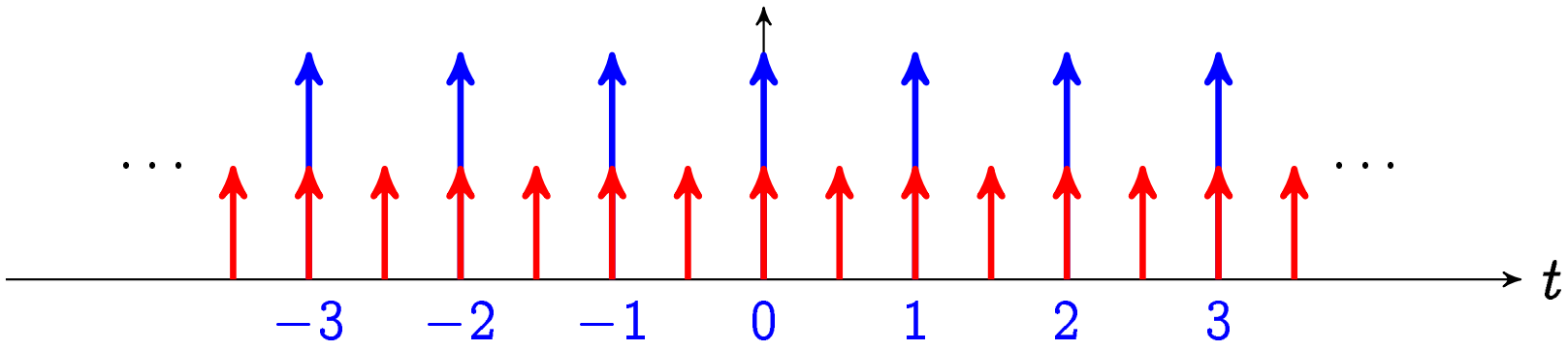
shah function

What is $ш(2t)$?

shah function

What is $\sqcup(2t)$?

$$\sqcup(2t) = \sum_{n=-\infty}^{\infty} \delta(2t - n) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta(t - \tfrac{n}{2}) \tag{50}$$



To make each impulse of unit strength, we need $2\sqcup(2t)$.

shah function

Some more properties (assume n is an integer):

- $\text{Ш}(at) = \frac{1}{|a|} \sum_{n=-\infty}^{\infty} \delta(t - \frac{n}{a})$
- $\text{Ш}(-t) = \text{Ш}(t)$
- $\text{Ш}(t+n) = \text{Ш}(t)$
- $\text{Ш}(t - \frac{1}{2}) = \text{Ш}(t + \frac{1}{2})$
- $\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \text{Ш}(t)dt = 1$
- $\text{Ш}(t) = 0$ for $t \neq n$

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cosine and sine

What is the Fourier transform of $\cos \pi t$?

$$\mathcal{F}\{\cos \pi t\} = \mathcal{F}\left\{\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right\} \tag{51}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{j\pi t} e^{-j2\pi st} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j\pi t} e^{-j2\pi st} dt \tag{52}$$

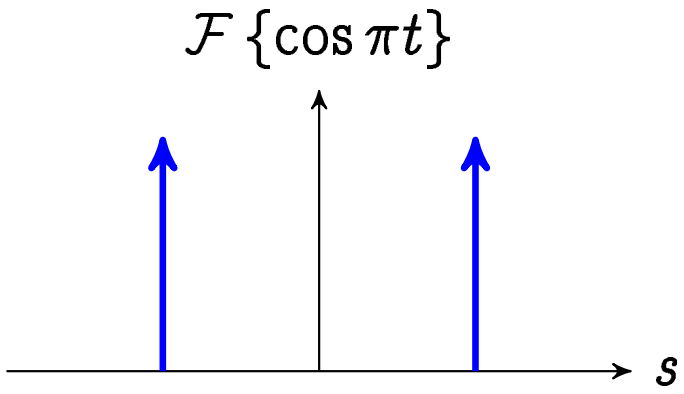
$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(s-\frac{1}{2})t} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(s+\frac{1}{2})t} dt \tag{53}$$

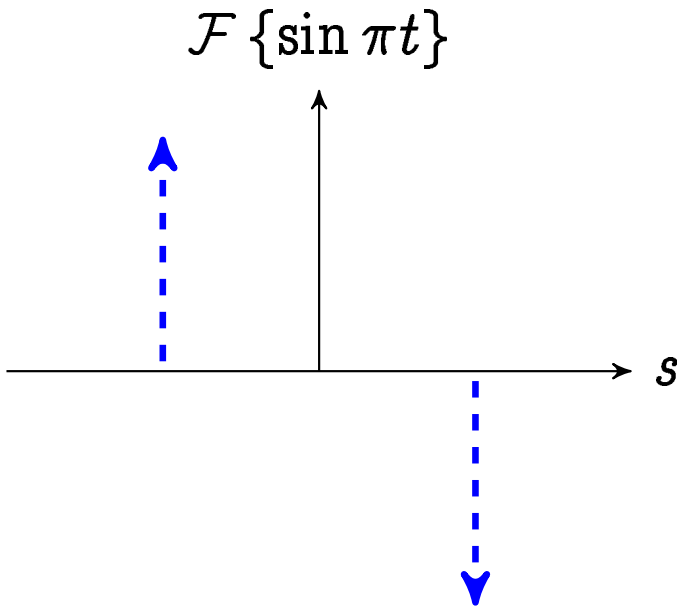
$$= \frac{1}{2}\delta(s - \tfrac{1}{2}) + \frac{1}{2}\delta(s + \tfrac{1}{2}) \tag{54}$$

cosine and sine

$$\cos \pi t \supset \frac{1}{2}\left[\delta(s-\tfrac{1}{2})+\delta(s+\tfrac{1}{2})\right] \tag{55}$$

$$\sin \pi t \supset \frac{j}{2}\left[\delta(s+\tfrac{1}{2})-\delta(s-\tfrac{1}{2})\right] \tag{56}$$





cosine and sine

The same "pair of impulses" exist for different sinusoidal frequencies, e.g.,

$$\cos t \supset \frac{1}{2}\left[\delta(s - \tfrac{1}{2\pi}) + \delta(s + \tfrac{1}{2\pi})\right] \tag{57}$$

$$\sin t \supset \frac{j}{2}\left[\delta(s + \tfrac{1}{2\pi}) - \delta(s - \tfrac{1}{2\pi})\right] \tag{58}$$

shah function

What is $\mathcal{F}\{\text{Ш}(t)\}$?

shah function

What is $\mathcal{F}\{\sqcup\!\sqcup(t)\}$?

Since $\mathcal{F}\{\delta(t-n)\} = \int_{-\infty}^{\infty} \delta(t-n)e^{-j2\pi st}dt = e^{-j2\pi ns}$, we have

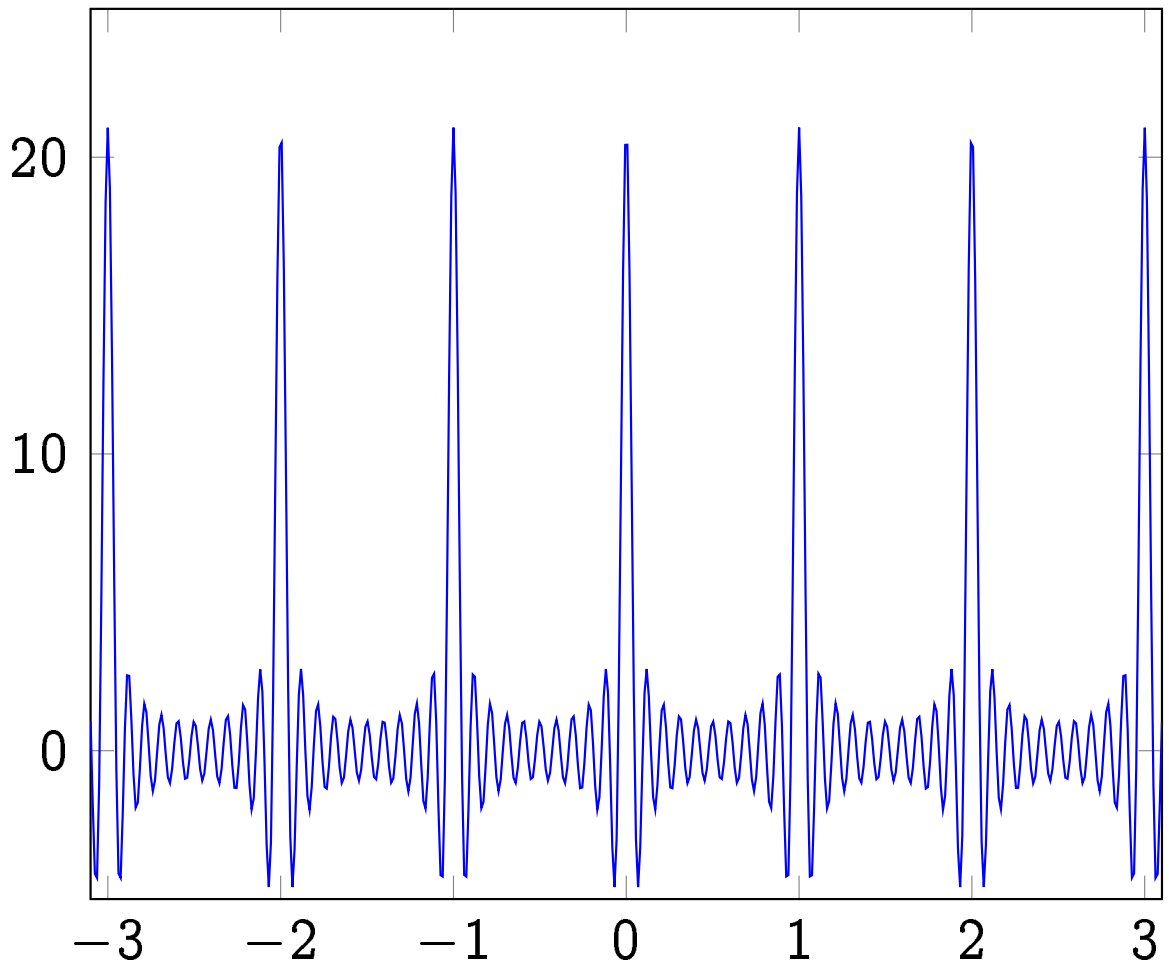
$$\mathcal{F}\{\sqcup\!\sqcup(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t-n)\right\} = \sum_{n=-\infty}^{\infty} e^{-j2\pi ns} \tag{59}$$

This function "blows up" when s takes on integer values, but is zero otherwise.

Is it $\sqcup\!\sqcup(s)$? Or a scaled version of it?

shah function

This is a plot of $\sum_{n=-10}^{10} e^{-j2\pi ns}$



shah function

We notice that $F(s) = \sum_{n=-\infty}^{\infty} e^{-j2\pi ns}$ is a *periodic function with period 1* since $F(s) = F(s+1) = F(s+2)$ etc.

We make use of Fourier series of a function $G(t)$ with period L:

$$G(s) = \sum_{n=-\infty}^{\infty} \alpha_n e^{j2\pi \frac{n}{L} s} \tag{60}$$

$$\alpha_n = \frac{1}{L} \int_{-L/2}^{L/2} G(s) e^{-j2\pi \frac{n}{L} s} ds \tag{61}$$

Now we consider $L = 1$.

shah function

- Since $F(s) = \sum_{n=-\infty}^{\infty} e^{-j2\pi ns} = \sum_{n=-\infty}^{\infty} e^{j2\pi ns}$, comparing with Eq. (60), we have

$$\alpha_n = 1 \tag{62}$$

- Let $\hat{F}(s) = F(s)\Pi(s)$, i.e., only one period. Eq. (61) now says:

$$1 = \int_{-1/2}^{1/2} F(s)e^{-j2\pi ns}ds \tag{63}$$

$$= \int_{-\infty}^{\infty} F(s)\Pi(s)e^{-j2\pi ns}ds \tag{64}$$

So $\hat{F}(s)$ is the (inverse) Fourier transform of 1, which means $\hat{F}(s) = \delta(s)$.

shah function

- We can write $F(s)$ as a summation of (shifted) $\hat{F}(s)$:

$$F(s) = \sum_{n=-\infty}^{\infty} \hat{F}(s-n) = \sum_{n=-\infty}^{\infty} \delta(s-n) = \text{ш}(s) \tag{65}$$

$$\boxed{\text{ш}(t) \supset \text{ш}(s)} \tag{66}$$