Fundamentals of Fourier transform

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ELEC8501: The Fourier Transform and Its Applications

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https://www.eee.hku.hk/~elec8501

1 The Fourier transform

2 Symmetry arguments and conjugate function

3 Visualizing Fourier transforms

4 Interpreting Fourier transforms

Equations

The (continuous) Fourier transform equations:

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \tag{1}$$

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{j2\pi st} ds \tag{2}$$

- Can write $F(s) = \mathcal{F}\{f(t)\}$ and $f(t) = \mathcal{F}^{-1}\{F(s)\}$
- Normally say $f(t)$ is in the *time domain*, $F(s)$ is in the *frequency domain* or *Fourier domain*
- Can also write $f(t) \supset F(s)$
- i or j? $F(s)$ or $\hat{f}(s)$?
- Use $\omega = 2\pi s$? Symmetry in the two equations?

Equations

More critical (mathematical) questions:

- Integral exists?
- How do I know

$$f(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \right] e^{j2\pi st} ds? \tag{3}$$

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"The question of the existence of transforms may safely be ignored when the function to be transformed is an accurately specified description of a physical quantity. Physical possibility is a valid sufficient condition for the existence of a transform." (Quote from Prof. Bracewell) — note that e.g., $f(t) = \sin t$ is not physical...

Related equations

Eq. (1) and (2) are known as continuous Fourier transform (CFT).

Discrete-time Fourier transform (DTFT):

$$\boxed{F(s) = \sum_{n=-\infty}^{\infty} f[n] e^{-j2\pi sn}} \tag{4}$$

Discrete Fourier transform (DFT):

$$\boxed{F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi kn/N}} \tag{5}$$

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Symmetry

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Any function $f(t)$ can be split unambiguously into odd and even parts:

$$f(t) = e(t) + o(t) \tag{8}$$

(how do you show that this decomposition is unique?)

Symmetry

Calculating $e(t)$ and $o(t)$ from $f(t)$:

$$e(t) = \frac{1}{2}\Big[f(t) + f(-t)\Big] \tag{9}$$

$$o(t) = \frac{1}{2}\Big[f(t) - f(-t)\Big] \tag{10}$$

(try some examples yourselves)

Symmetry

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \tag{11}$$

$$= \int_{-\infty}^{\infty} e(t) \cos(2\pi st)\, dt - j \int_{-\infty}^{\infty} e(t) \sin(2\pi st)\, dt$$

$$+ \int_{-\infty}^{\infty} o(t) \cos(2\pi st)\, dt - j \int_{-\infty}^{\infty} o(t) \sin(2\pi st)\, dt \tag{12}$$

$$= \int_{-\infty}^{\infty} e(t) \cos(2\pi st)\, dt - j \int_{-\infty}^{\infty} o(t) \sin(2\pi st)\, dt \tag{13}$$

If $f(t)$ is even? Or $f(t)$ is real and even? Or $f(t)$ is real?

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$$\text{even} \supset \text{even} \tag{14}$$

$$\text{odd} \supset \text{odd} \tag{15}$$

Symmetry

Breaking down further with real and imaginary parts:

$$f(t) = \mathrm{Re}\{e(t)\} + j\mathrm{Im}\{e(t)\} + \mathrm{Re}\{o(t)\} + j\mathrm{Im}\{o(t)\}$$

$$F(s) = \mathrm{Re}\{E(s)\} + j\mathrm{Im}\{E(s)\} + \mathrm{Re}\{O(s)\} + j\mathrm{Im}\{O(s)\}$$

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$$\text{real and even} \supset \text{real and even} \tag{16}$$

$$\text{imaginary and even} \supset \text{imaginary and even} \tag{17}$$

$$\text{real and odd} \supset \text{imaginary and odd} \tag{18}$$

$$\text{imaginary and odd} \supset \text{real and odd} \tag{19}$$

Conjugate

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$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \tag{20}$$

$$F^*(s) = \int_{-\infty}^{\infty} f^*(t) e^{j2\pi st} dt \tag{21}$$

$$F^*(-s) = \int_{-\infty}^{\infty} f^*(t) e^{-j2\pi(-s)t} dt \tag{22}$$

So, in general,

$$\boxed{f^*(t) \supset F^*(-s)} \tag{23}$$

Conjugate

Special case: $f(t)$ is real.

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If $f(t)$ is real, then $f(t) = f^*(t)$, and therefore $F(s) = F^*(-s)$. The real part therefore is even, and the imaginary part is odd. We say that such a function is *Hermitian*.

$$\text{real} \supset \text{Hermitian} \tag{24}$$

$$\text{imaginary} \supset \text{anti-Hermitian} \tag{25}$$

Conjugate

What does $f(t) + f^*(t)$ transform to?

Conjugate

What does $f(t) + f^*(t)$ transform to?

$f(t) + f^*(t) \supset F(s) + F^*(-s)$. Therefore,

$$\boxed{\mathrm{Re}\{f(t)\} \supset \frac{1}{2}\left(F(s) + F^*(-s)\right)} \tag{26}$$

How about $\mathrm{Im}\{f(t)\}$?

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Mental picture

You can use a computer to calculate (numerical approximation of) various Fourier transforms. So why do we bother with:

- doing it by "paper and pencil"?
- doing it mentally?

Example 1

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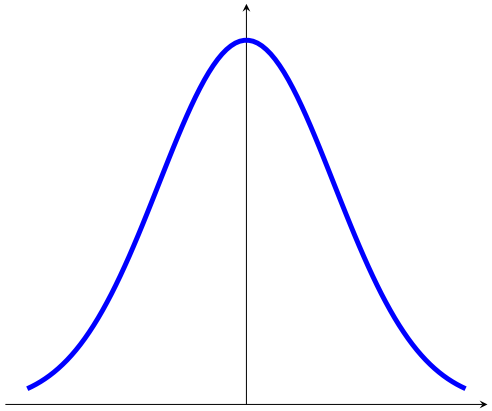
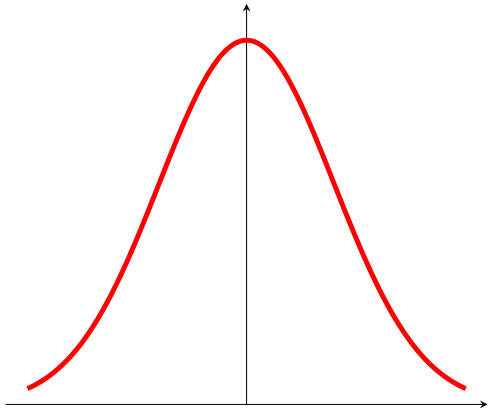
ANS: It is a real and even function, so its Fourier transform is also real and even. We'll have to "do the math" to get to the actual answer, which is $F(s) = e^{-\pi s^2}$. But because both domains are real (only), we can easily plot and think about them:

$f(t) = e^{-\pi t^2}$



$F(s) = e^{-\pi s^2}$



Example 2

What is the Fourier transform of $f(t) = te^{-\pi t^2}$?

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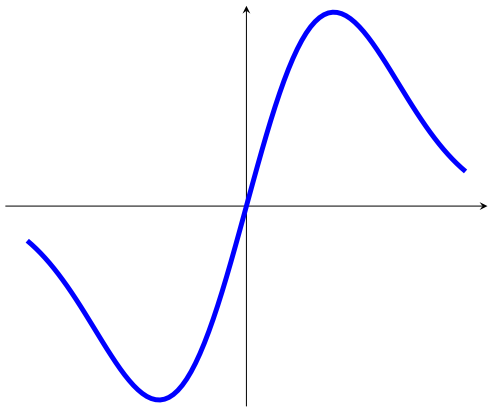
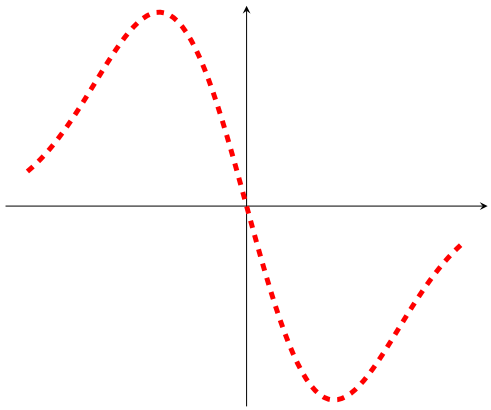
What is the Fourier transform of $f(t) = te^{-\pi t^2}$?

ANS: It is a real and odd function, so its Fourier transform is imaginary and odd. We can plot only the imaginary part in the Fourier transform:

$f(t) = te^{-\pi t^2}$



$F(s) = -jse^{-\pi s^2}$



Example 3

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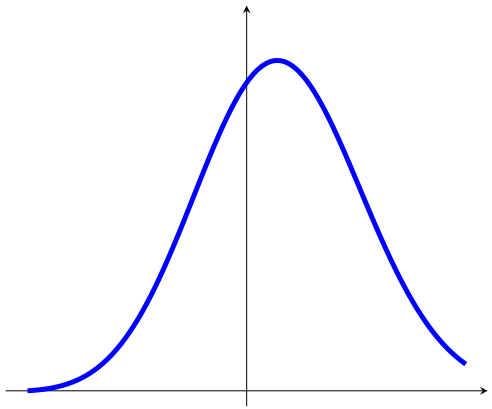
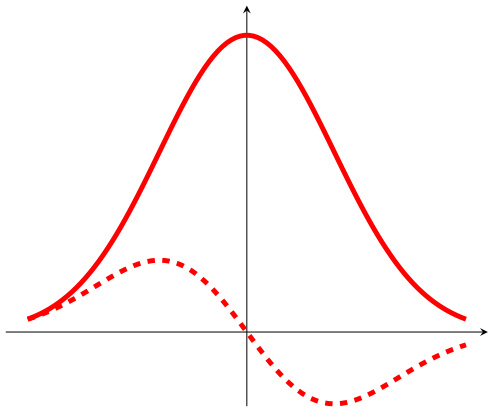
ANS: It is a real function (but neither odd nor even), so its Fourier transform is Hermitian (complex). We can *choose* to plot both the real and imaginary parts on the same diagram: (Sometimes, we may choose to plot only the magnitude)

$f(t) = (1+t)e^{-\pi t^2}$



$F(s) = (1-js)e^{-\pi s^2}$



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Meaning of Fourier transform

Remember from Eq. (2) that

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{j2\pi st} ds \tag{27}$$

So, we *decompose* $f(t)$ into a sum (integration) of exponentials (sines and cosines at various frequencies), where each $F(s)$ serves as the *weight*

Therefore, $F(s)$ gives us a frequency (spectrum) view of the signal