

Homework 3

{ Due: Nov 22, 2023, 11:55pm }

Instruction: Submit your answers electronically through Moodle. In Moodle, you must submit under the PDF Submission link with ONE PDF file containing answers to all remaining questions. Please ensure that you use notation symbols in alignment with the lecture notes.

Question 1

Suppose the function $f(t)$ is zero outside the interval $-\frac{1}{2} \leq t \leq \frac{1}{2}$. We form a function $g(t)$ which is a periodic version of $f(t)$ with period 1 by the formula

$$g(t) = \sum_{k=-\infty}^{\infty} f(t - k) \quad (1)$$

The Fourier series representation of $g(t)$ is given by

$$g(t) = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{j2\pi n t}. \quad (2)$$

where $\hat{g}(n)$ is the Fourier coefficients of the periodic function $g(t)$. Prove that $\hat{g}(n)$ are the samples of the Fourier Transform of $f(t)$, evaluated at the integer values.

Question 2

Let $F(s) = \mathcal{F}\{f(t)\}$ and $G(s) = \frac{1}{j2\pi} \mathcal{F}\{f'(t)\} = sF(s)$. Suppose that the signal $f(t)$ is band-limited with $F(s) = 0$ for $|s| > 1$.

1. Show that for all s ,

$$F(s) = \Lambda(s) (\mathbf{W}(s) * F(s)) - \Lambda'(s) (\mathbf{W}(s) * G(s)), \quad (3)$$

where $\Lambda(s)$ is the triangle function

$$\Lambda(s) = \begin{cases} 1 - |s|, & |s| \leq 1, \\ 0, & |s| > 1. \end{cases} \quad (4)$$

2. Inspired by the previous derivation, can we interpolate the values $f(t)$ (bandwidth = 2) for all t at a sampling rate of 1, using the signal samples and their derivative values (i.e., $f(n)$ and $f'(n)$ for all integers n)? If possible, please provide the interpolation formula. (Hint: You may consider the inverse Fourier transform of $F(s)$.)

Question 3

Using the Abel-Hankel-Fourier ring, prove the Abel transforms as follows:

1. $\mathcal{A}\{\text{jinc}^2(r)\} = \text{sinc}^2(x)$

2. $\mathcal{A}\left\{\frac{\delta(r)}{r}\right\} = 2\pi\delta(x)$

$$3. \mathcal{A} \left\{ e^{-\pi r^2} \right\} = e^{-\pi x^2}$$

$$4. \mathcal{A} \{ 2\pi a J_0(2\pi ar) \} = 2 \cos(2\pi ax)$$

Hint: The Bessel orthogonality theorem is described as:

$$\int_0^\infty ax J_0(ax) J_0(bx) dx = \begin{cases} 1, & |b| = a \\ 0, & |b| \neq a \end{cases} \quad (5)$$

Question 4

The Cauchy random variable X has the following probability density distribution

$$f_X(x) = \frac{1}{\pi(1+x^2)}. \quad (6)$$

Proof the following statements:

1. The reciprocal of the Cauchy random variable (i.e., $Y = \frac{1}{X}$) has the same Cauchy density distribution.
2. The distribution of the average of two (independent) Cauchy distribution random variables (i.e., $Y = \frac{1}{2}(X_1 + X_2)$, where X_1 and X_2 are distributed as above) has the same Cauchy density distribution.

You are given the Fourier pair $\frac{2a}{a^2 + 4\pi^2 x^2} \supset e^{-a|s|}$, and the fact that if a random variable X has the probability density distribution $f_X(x)$ then its linear transformation $a + bX$ has the density distribution $\frac{1}{|b|} f_X\left(\frac{x-a}{b}\right)$.

Question 5

The *average operator* depending on a parameter $\tau > 0$ is defined as

$$\mathcal{A}_\tau f(x) = \frac{1}{2\tau} \int_{x-\tau}^{x+\tau} f(t) dt, \quad (7)$$

where $f(t)$ is a periodic signal of period 1. The Fourier series for $f(t)$ is expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n t}, \quad (8)$$

where c_n is the Fourier coefficient. Show that the new signal $\mathcal{A}_\tau f(x)$ is also periodic of period 1 by finding its Fourier series in terms of the Fourier series for $f(t)$.

Question 6

An LTI (Linear Time-Invariant) system is defined as a system that satisfies the properties of linearity and time-invariance with a block diagram shown in figure 1.

To be specific, let $a(t)$ be the input signal, $b(t)$ be the output signal, a LTI system L should satisfy:

- Linearity: A system L is considered linear if it satisfies the equation:

$$L\{\alpha_1 a_1(t) + \alpha_2 a_2(t)\} = \alpha_1 L\{a_1(t)\} + \alpha_2 L\{a_2(t)\}, \quad (9)$$

where α_1 and α_2 are constants, and $a_1(t)$ and $a_2(t)$ are input signals.

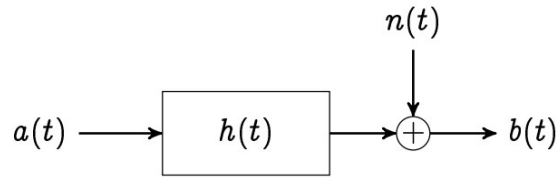


Figure 1: The block diagram of an LTI. Herein, $a(t)$ is the input signal, $b(t)$ is the output signal, and $h(t)$ is the impulse response. We consider an ideal LTI with $n(t) = 0$.

- Time-invariant: Given that $b(t) = L\{a(t)\}$, a system L is considered time-invariant if it satisfies the equation:

$$b(t - \tau) = L\{a(t - \tau)\}. \quad (10)$$

Now let's define another version of the *average operator* in the context of LTI. Let $c > 0$ and the (continuous) running average of an input signal $a(t)$ can be defined to be

$$b(t) = L\{a(t)\} = \frac{1}{2c} \int_{t-c}^{t+c} a(x) dx \quad (11)$$

which is defined in Q3.

1. Show that L is an LTI system.
2. Find its impulse response $h(t)$ and transfer function $H(s)$ as is shown in figure 1, with $h(t) \supset H(s)$.
3. What happens to the system as $c \rightarrow 0$? Justify your answer based on the properties of the impulse response.