

HomeworkAns 3

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Question 1

To calculate the Fourier series coefficient, we can take the integral over any period of the signal. Suppose that we pick the period $(-\frac{1}{2}, \frac{1}{2})$, the Fourier coefficients of $g(t)$ are

$$\hat{g}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t) e^{-j2\pi n t} dt. \quad (1)$$

Since $f(t)$ is zero outside the interval, and $g(t)$ is the replicas of $f(t)$ with period 1, both functions are equal in the interval. Thus,

$$\hat{g}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) e^{-j2\pi n t} dt \quad (2)$$

$$= \int_{-\infty}^{\infty} f(t) e^{-j2\pi n t} dt, \quad (3)$$

which is indeed the Fourier transform of $f(t)$. Therefore, we get

$$\hat{g}(n) = F(s) |_{s=n}. \quad (4)$$

The Fourier coefficients of the periodic function $g(t)$ are the sample of the Fourier Transform of $f(t)$, taken all the integers.

Question 2

1. For $0 \leq s \leq 1$, we only get a few shifts $F(s - n)$ that are non-zero, namely

$$\mathfrak{W}(s) * F(s) = \sum_{n=-\infty}^{\infty} F(s - n) \quad (5)$$

$$= \underbrace{\cdots + F(s + 2) + F(s + 1)}_{=0} + F(s) + F(s - 1) + \underbrace{F(s - 2) + \cdots}_{=0} \quad (6)$$

$$= F(s) + F(s - 1). \quad (7)$$

Since $G(s) = sF(s)$,

$$\mathfrak{W}(s) * G(s) = \sum_{n=-\infty}^{\infty} (s - n)F(s - n) = sF(s) + (s - 1)F(s - 1). \quad (8)$$

Multiplying Eq.(7) by $(s - 1)$ and subtract Eq.(8), it gives

$$(s - 1)(\mathfrak{W}(s) * F(s)) - \mathfrak{W}(s) * G(s) = -F(s), \quad (9)$$

or

$$F(s) = (1 - s)(\mathfrak{W}(s) * F(s)) + \mathfrak{W}(s) * G(s). \quad (10)$$

For $-1 \leq s < 0$,

$$\mathfrak{W}(s) * F(s) = \sum_{n=-\infty}^{\infty} F(s-n) = F(s+1) + F(s), \quad (11)$$

and

$$\mathfrak{W}(s) * G(s) = \sum_{n=-\infty}^{\infty} (s-n)F(s-n) = (s+1)F(s+1) + sF(s). \quad (12)$$

Multiplying the Eq.(11) by $(s+1)$ and subtract Eq.(12), it gives

$$F(s) = (s+1) (\mathfrak{W}(s) * F(s)) - \mathfrak{W}(s) * G(s). \quad (13)$$

Combine Eq.(10) and Eq.(13), Since

$$\Lambda(s) = \begin{cases} 1-|s|, & |s| \leq 1 \\ 0, & |s| > 1 \end{cases} = \begin{cases} 1-s, & 0 \leq s \leq 1 \\ 1+s, & -1 \leq s \leq 0 \\ 0, & |s| > 1 \end{cases}, \quad (14)$$

and

$$\Lambda'(s) = \begin{cases} -1, & 0 \leq s \leq 1 \\ 1, & -1 \leq s \leq 0 \\ 0, & |s| > 1 \end{cases}, \quad (15)$$

Combine Eq.(10) and Eq.(13), we can write

$$F(s) = \Lambda(s) (\mathfrak{W}(s) * F(s)) - \Lambda'(s) (\mathfrak{W}(s) * G(s)) \quad (16)$$

for all $-\infty < s < \infty$.

2. We first take the inverse Fourier transform of the first term on the right in Eq.(16)

$$\mathcal{F}^{-1} \{ \Lambda(s) (\mathfrak{W}(s) * F(s)) \} = \text{sinc}^2(t) * (f(t) \mathfrak{W}(t)) \quad (17)$$

$$= \text{sinc}^2(t) * \left(\sum_{n=-\infty}^{\infty} f(n) \delta(t-n) \right) \quad (18)$$

$$= \sum_{n=-\infty}^{\infty} f(n) \text{sinc}^2(t-n). \quad (19)$$

For the second term, the inverse Fourier transform gives

$$\mathcal{F}^{-1} \{ \Lambda'(s) (\mathfrak{W}(s) * G(s)) \} = -j2\pi t \text{sinc}^2(t) * \left(\frac{1}{j2\pi} f'(t) \mathfrak{W}(t) \right) \quad (20)$$

$$= - (t \text{sinc}^2(t)) * \left(\sum_{n=-\infty}^{\infty} f'(n) \delta(t-n) \right) \quad (21)$$

$$= - \sum_{n=-\infty}^{\infty} f'(n) (t-n) \text{sinc}^2(t-n). \quad (22)$$

Adding these two results together we obtain the interpolation formula

$$f(t) = \mathcal{F}^{-1} \{ \Lambda(s) (\mathfrak{W}(s) * F(s)) \} - \mathcal{F}^{-1} \{ \Lambda'(s) (\mathfrak{W}(s) * G(s)) \} \quad (23)$$

$$= \sum_{n=-\infty}^{\infty} f(n) \text{sinc}^2(t-n) + \sum_{n=-\infty}^{\infty} f'(n) (t-n) \text{sinc}^2(t-n). \quad (24)$$

Question 3

1. $\mathcal{A} \{ \text{jinc}^2(r) \} = \text{sinc}^2(x)$

The Hankel transform of the $\text{jinc}^2(r)$ is:

$$\mathcal{B} \{ \text{jinc}^2(r) \} = \mathcal{B} \{ \text{jinc}(r) \text{jinc}(r) \} \quad (25)$$

$$= \mathcal{B} \{ \text{jinc}(r) \} * \mathcal{B} \{ \text{jinc}(r) \} \quad (26)$$

$$= \Pi(\rho) * \Pi(\rho), \quad (27)$$

where $*$ represents the convolution operator. From the Abel-Hankel-Fourier ring theorem, the Abel transform of $\text{jinc}^2(r)$ should be the inverse Fourier transform of the slice: $\Pi(s) * \Pi(s) = \Lambda(s)$. Therefore, it is easy to have:

$$\mathcal{F}^{-1} \{ \Lambda(s) \} = \text{sinc}^2(x) = \mathcal{A} \{ \text{jinc}^2(r) \}. \quad (28)$$

Q.E.D.

2. $\mathcal{A} \left\{ \frac{\delta(r)}{r} \right\} = 2\pi\delta(x)$

From HW2 Q1(2), we can derive the Bessel transform of $\frac{\delta(r)}{r} \supset 2\pi$, i.e.,

$$\mathcal{B} \left\{ \frac{\delta(r)}{r} \right\} = 2\pi \int_0^\infty \frac{\delta(r)}{r} J_0(2\pi r \rho) r dr \quad (29)$$

$$= 2\pi \int_0^\infty \delta(r) J_0(2\pi r \rho) dr \quad (30)$$

$$= 2\pi J_0(0) = 2\pi. \quad (31)$$

From the Abel-Hankel-Fourier ring theorem, the Abel transform of $\frac{\delta(r)}{r}$ should be the inverse Fourier transform of the slice: 2π . Therefore, it is easy to have:

$$\mathcal{F}^{-1} \{ 2\pi \} = 2\pi\delta(x) = \mathcal{A} \left\{ \frac{\delta(r)}{r} \right\}. \quad (32)$$

Q.E.D.

3. $\mathcal{A} \left\{ e^{-\pi r^2} \right\} = e^{-\pi x^2}$

First, we derive the Bessel transform of $e^{-\pi r^2}$. Since $e^{-\pi r^2} = e^{-\pi(x^2+y^2)}$, we have:

$$\mathcal{B} \left\{ e^{-\pi r^2} \right\} = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-\pi(x^2+y^2)} e^{-j2\pi(ux+vy)} dx dy \quad (33)$$

$$= \left(\int_{-\infty}^\infty e^{-\pi x^2} e^{-j2\pi ux} dx \right) \left(\int_{-\infty}^\infty e^{-\pi y^2} e^{-j2\pi vy} dy \right) \quad (34)$$

$$= \mathcal{F} \left\{ e^{-\pi x^2} \right\} \mathcal{F} \left\{ e^{-\pi y^2} \right\} \quad (35)$$

$$= e^{-\pi(u^2+v^2)} \quad (36)$$

$$= e^{-\pi \rho^2}. \quad (37)$$

From the Abel-Hankel-Fourier ring theorem, the Abel transform of $e^{-\pi r^2}$ should be the inverse Fourier transformation of the slice: $e^{-\pi s^2}$. Therefore, it is easy to have:

$$\mathcal{F}^{-1} \left\{ e^{-\pi s^2} \right\} = e^{-\pi x^2} = \mathcal{A} \left\{ e^{-\pi r^2} \right\}. \quad (38)$$

Q.E.D.

4. $\mathcal{A}\{2\pi a J_0(2\pi ar)\} = 2 \cos(2\pi ax)$

The Bessel transform of $2\pi a J_0(2\pi ar)$ can be calculated as:

$$\mathcal{B}\{2\pi a J_0(2\pi ar)\} = 2\pi \int_0^\infty (2\pi ar) J_0(2\pi ar) J_0(2\pi r\rho) dr \quad (39)$$

$$= \int_0^\infty a \hat{r} J_0(a \hat{r}) J_0(\rho \hat{r}) d\hat{r}, \quad (40)$$

where $\hat{r} = 2\pi r$.

Given the Bessel orthogonality theorem:

$$\int_0^\infty ax J_0(ax) J_0(bx) dx = \begin{cases} 1, & |b| = a \\ 0, & |b| \neq a \end{cases}. \quad (41)$$

Therefore, by substituting $\hat{r}$ as x into equation (41), we have the equation (40) equals to

$$\delta(\rho - a) + \delta(\rho + a). \quad (42)$$

From the Abel-Hankel-Fourier ring theorem, the Abel transform of $2\pi a J_0(2\pi ar)$ should be the inverse Fourier transformation of the slice: $\delta(s - a) + \delta(s + a)$. Therefore, it is easy to have:

$$\mathcal{F}^{-1}\{\delta(s - a) + \delta(s + a)\} = e^{-j2\pi ax} + e^{j2\pi ax} = 2 \cos(2\pi ax) = \mathcal{A}\{2\pi a J_0(2\pi ar)\}. \quad (43)$$

Q.E.D.

Question 4

1. According to the definition of the characteristic function

$$F_Y(s) = \mathbb{E}(e^{jsy}) = \mathbb{E}(e^{js/x}) = \int_{-\infty}^\infty e^{js/x} f_X(x) dx = \int_{-\infty}^\infty \frac{e^{js/x}}{\pi(1+x^2)} dx. \quad (44)$$

Substitute $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2} dy$, we have

$$F_Y(s) = \int_{-\infty}^\infty \frac{e^{jsy}}{\pi(1+y^{-2})} \cdot \frac{1}{y^2} dy = \int_{-\infty}^\infty \frac{e^{jsy}}{\pi(1+y^2)} dy. \quad (45)$$

Therefore, the random variable Y has the same Cauchy density.

2. The sum we want is $\frac{1}{2}X_1 + \frac{1}{2}X_2$, where each term has the distribution

$$2f_X(2x) = \frac{2}{\pi(1+4x^2)}. \quad (46)$$

Using the provided Fourier pair, the corresponding characteristic function for each term is given by

$$\mathcal{F}\left\{\frac{2}{\pi(1+4x^2)}\right\} = \mathcal{F}\left\{\frac{2\pi}{\pi^2 + 4\pi^2 t^2}\right\} = e^{-\pi|s|}. \quad (47)$$

The characteristic function of the random variable Y is the multiplication of two characteristic functions (Property in Lec7 Page35) as:

$$F_Y(s) = e^{-\pi|s|} \cdot e^{-\pi|s|} = e^{-2\pi|s|}. \quad (48)$$

Finally, the distribution is its inverse Fourier transform

$$f_Y(y) = \mathcal{F}^{-1}\{e^{-2\pi|s|}\} = \frac{4\pi}{4\pi^2 + 4\pi^2 y^2} = \frac{1}{\pi(1+y^2)}, \quad (49)$$

which is exactly the same Cauchy distribution.

Question 5

$$\mathcal{A}_\tau f(x) = \frac{1}{2\tau} \int_{x-\tau}^{x+\tau} f(t) dt \quad (50)$$

$$= \frac{1}{2\tau} \int_{x-\tau}^{x+\tau} \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n t} dt \quad (51)$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{2\tau} \int_{x-\tau}^{x+\tau} e^{j2\pi n t} dt. \quad (52)$$

Therefore, it is the integral we have to evaluate.

If $n = 0$, we have

$$\frac{1}{2\tau} \int_{x-\tau}^{x+\tau} 1 dt = \frac{1}{2\tau} ((x+\tau) - (x-\tau)) = 1. \quad (53)$$

If $n \neq 0$,

$$\frac{1}{2\tau} \int_{x-\tau}^{x+\tau} e^{j2\pi n t} dt = \frac{1}{2\tau} \left[\frac{1}{j2\pi n} e^{j2\pi n t} \right]_{t=x-\tau}^{t=x+\tau} \quad (54)$$

$$= \frac{1}{j4\pi n \tau} (e^{j2\pi n(x+\tau)} - e^{j2\pi n(x-\tau)}) \quad (55)$$

$$= \frac{1}{j4\pi n \tau} e^{j2\pi n x} (e^{j2\pi n \tau} - e^{-j2\pi n \tau}) \quad (56)$$

$$= \frac{1}{2\pi n \tau} e^{j2\pi n x} \frac{e^{j2\pi n \tau} - e^{-j2\pi n \tau}}{2j} \quad (57)$$

$$= \frac{\sin 2\pi n \tau}{2\pi n \tau} e^{j2\pi n x} \quad (58)$$

$$= \text{sinc}(2\pi n \tau) e^{j2\pi n x}. \quad (59)$$

Therefore, the Fourier series of $\mathcal{A}_\tau f(x)$ is

$$\mathcal{A}_\tau f(x) = \sum_{n=-\infty}^{\infty} c_n \text{sinc}(2\pi n \tau) e^{j2\pi n x}, \quad (60)$$

which is a periodic signal of period 1.

Question 6

1. First let's show that L is linear:

$$L(\alpha_1 a_1(t) + \alpha_2 a_2(t)) = \frac{1}{2c} \int_{t-c}^{t+c} (\alpha_1 a_1(x) + \alpha_2 a_2(x)) dx \quad (61)$$

$$= \alpha_1 \frac{1}{2c} \int_{t-c}^{t+c} a_1(x) dx + \alpha_2 \frac{1}{2c} \int_{t-c}^{t+c} a_2(x) dx \quad (62)$$

$$= \alpha_1 L(a_1(t)) + \alpha_2 L(a_2(t)). \quad (63)$$

Hence, L is linear.

Now let's show that L is time-invariant:

$$L\{a(t-\tau)\} = \frac{1}{2c} \int_{t-c}^{t+c} a(x-\tau) dx \quad (64)$$

$$= \frac{1}{2c} \int_{(t-\tau)-c}^{(t-\tau)+c} a(y) dy \quad (65)$$

$$= b(t-\tau), \quad (66)$$

where we can recognize $y = t - \tau$. Hence, we have shown that L is indeed time-invariant.

2. If we define

$$h(t) = \frac{1}{2c} \Pi \left(\frac{t}{2c} \right), \quad (67)$$

then clearly $L\{a(t)\} = h(t) * a(t)$. Hence, the impulse response of the system is given by $h(t)$, and the transfer function is simply:

$$H(s) = \text{sinc}(2cs). \quad (68)$$

3. If we let $c \rightarrow 0$, then the impulse response function becomes:

$$h(t) = \lim_{c \rightarrow 0} \frac{1}{2c} \Pi \left(\frac{t}{2c} \right) = \delta(t). \quad (69)$$

In the frequency domain, the transfer function becomes:

$$H(s) = \lim_{c \rightarrow 0} \text{sinc}(2cs) = 1, \quad (70)$$

which makes perfect sense for an impulse function.