

Homework 2

{ Due: Oct 31, 2023, 11:55pm }

Instruction: Submit your answers electronically through Moodle. In Moodle, you must submit under the PDF Submission link with ONE PDF file containing answers to all remaining questions. Please ensure that you use notation symbols in alignment with the lecture notes.

Question 1

Prove the Fourier-Bessel transforms pair of

$$1. \mathcal{B} \{ r^{-1} e^{-ar} \} = \frac{2\pi}{\sqrt{a^2 + (2\pi\rho)^2}}$$

$$2. \mathcal{B} \left\{ \frac{\delta(r)}{r} \right\} = 2\pi$$

$$3. \mathcal{B} \{ H(a-r) \} = \frac{a}{\rho} J_1(2\pi a\rho),$$

where $H(r)$ is the Heaviside unit step function.

Hint: Using the mathematical identity in Q1:

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta} d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}} \quad (1)$$

Question 2

Evaluate the following integrals using Rayleigh's theorem

$$1. \int_{-\infty}^{\infty} \frac{2}{1+(2\pi t)^2} \text{sinc}(2t) dt$$

$$2. \int_{-\infty}^{\infty} t^2 \text{sinc}^4(t) dt$$

Question 3

1. If $f(t) * g(t) = h(t)$, what is $f(t - \tau) * g(t + \tau)$ in terms of $h(t)$? (* stands for convolution)

2. Based on the previous derivation, show that the following relation holds

$$\int_{-\infty}^{\infty} f(t)g(-t)dt = \int_{-\infty}^{\infty} F(s)G(s)ds, \quad (2)$$

where $F(s)$ and $G(s)$ denote the Fourier transform of $f(t)$ and $g(t)$, respectively.

Question 4

One of the most common use of convolution and the Fourier transform is in solving differential equations. Consider the following classic second order differential equation

$$\frac{d^2 u(t)}{dt^2} - u(t) = -f(t). \quad (3)$$

If $f(t)$ is a given function, please find $u(t)$. (Hint: You may find it helpful to consider the Fourier transform of $e^{-|t|}$)

Question 5

We define the moment of order n of a function as:

$$m_n = \int_{-\infty}^{\infty} t^n f(t) dt \quad (4)$$

Let $f(t)$ be a signal with Fourier transform $F(s)$.

1. Demonstrate that the moment of order n appears as:

$$m_n = \left(\frac{j}{2\pi}\right)^n F^{[n]}(0). \quad (5)$$

where $F^{[n]}(s) = \frac{d^n F(s)}{ds^n}$ refers to the n^{th} order derivative of $F(s)$.

2. Demonstrate that the decomposition of $F(s)$ into a Taylor series appears as follows:

$$F(s) = \sum_{n=0}^{\infty} (-j2\pi)^n m_n \frac{s^n}{n!} \quad (6)$$

Question 6

Consider a function $f(t)$ made up of a succession of line segments with no discontinuation, shown in Figure 1.

1. Write the mathematical expression of the first and the second derivative of $f(t)$.
2. Let $F(s)$ be the Fourier transform of $f(t)$, prove that its Fourier transform can be written as:

$$F(s) = -\frac{1}{4\pi^2 s^2} \left[\left(\frac{a_1}{t_1} \right) + \left(\frac{a_2 - a_1}{t_2 - t_1} - \frac{a_1}{t_1} \right) e^{-j2\pi t_1 s} + \left(\frac{a_3 - a_2}{t_3 - t_2} - \frac{a_2 - a_1}{t_2 - t_1} \right) e^{-j2\pi t_2 s} \right. \\ \left. + \left(\frac{a_4 - a_3}{t_4 - t_3} - \frac{a_3 - a_2}{t_3 - t_2} \right) e^{-j2\pi t_3 s} - \left(\frac{a_4 - a_3}{t_4 - t_3} \right) e^{-j2\pi t_4 s} \right] \quad (7)$$

Hint: Use the Fourier derivative theorem.

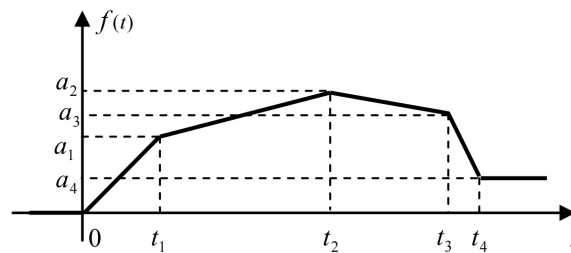


Figure 1: A piece-wise function.

Question 7

Use Generalized Fourier transforms to answer the below Filter questions. (Simplify your expression as much as possible.)

1. Find the Fourier transform of Filter 1 shown in Figure 2.
2. Find the Fourier transform of Filter 2 with the mathematical expression of $\sin(2\pi|t|)$, depicted in Figure 3.

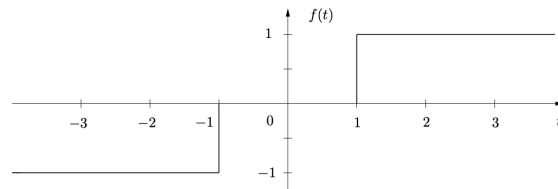


Figure 2: Filter 1.

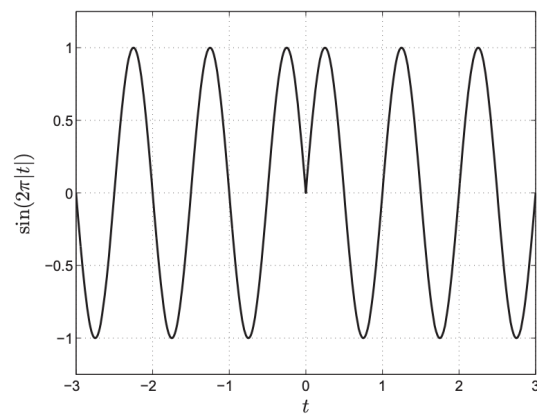


Figure 3: Filter 2.