

HomeworkAns 2

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Question 1

1.

$$\mathcal{B}\{r^{-1}e^{-ar}\} = 2\pi \int_0^\infty r^{-1}e^{-ar}J_0(2\pi r\rho)rdr \quad (1)$$

$$= 2\pi \int_0^\infty e^{-ar} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-j2\pi r\rho \cos\theta} d\theta \right) dr \quad (2)$$

$$= \int_0^{2\pi} \left(\int_0^\infty e^{-r(a+j2\pi\rho\cos\theta)} dr \right) d\theta \quad (3)$$

$$= \int_0^{2\pi} \frac{e^{-r(a+j2\pi\rho\cos\theta)}}{-(a+j2\pi\rho\cos\theta)} \Big|_0^\infty d\theta, \quad (4)$$

$$= \int_0^{2\pi} \frac{1}{a+j2\pi\rho\cos\theta} d\theta. \quad (5)$$

Then, we can use the given Mathematical identity

$$\int_0^{2\pi} \frac{1}{a+b\cos\theta} d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad (6)$$

where the original formula is the special case of $b = j2\pi\rho$. Therefore, it can be further written as

$$\mathcal{B}\{r^{-1}e^{-ar}\} = \frac{2\pi}{\sqrt{(a^2 - (j2\pi\rho)^2)}}, \quad (7)$$

$$= \frac{2\pi}{\sqrt{(a^2 + (2\pi\rho)^2)}}. \quad (8)$$

Q.E.D.

2.

$$\mathcal{B}\left\{\frac{\delta(r)}{r}\right\} = 2\pi \int_0^\infty \frac{\delta(r)}{r} J_0(2\pi r\rho)rdr \quad (9)$$

$$= 2\pi \int_0^\infty \delta(r) J_0(2\pi r\rho)dr \quad (10)$$

$$= 2\pi J_0(0) = 2\pi \quad (11)$$

3.

$$\mathcal{B}\{H(a-r)\} = 2\pi \int_0^\infty H(a-r) J_0(2\pi r\rho)rdr \quad (12)$$

$$= 2\pi \int_0^a J_0(2\pi r\rho)rdr \quad (13)$$

Let $\hat{\rho} = 2\pi r\rho$, so we have:

$$\mathcal{B}\{H(a-r)\} = \frac{2\pi}{(2\pi\rho)^2} \int_0^{a2\pi\rho} J_0(\hat{\rho})\hat{\rho}d\hat{\rho} \quad (14)$$

Use $\int_0^x r J_0(r) dr = x J_1(x)$, therefore:

$$\mathcal{B}\{H(a-r)\} = \frac{2\pi}{(2\pi\rho)^2} (2\pi a\rho) J_1(2\pi a\rho) \quad (15)$$

$$= \frac{a}{\rho} J_1(2\pi a\rho) \quad (16)$$

Question 2

1.

$$\int_{-\infty}^{\infty} \frac{2}{1+(2\pi t)^2} \text{sinc}(2t) dt = \int_{-\infty}^{\infty} \frac{2}{1+(2\pi t)^2} \text{sinc}^*(2t) dt \quad (17)$$

$$= \int_{-\infty}^{\infty} e^{-|s|} \frac{1}{2} \Pi^*\left(\frac{s}{2}\right) ds \quad (18)$$

$$= \int_{-1}^1 \frac{1}{2} e^{-|s|} ds \quad (19)$$

$$= 2 \int_0^1 \frac{1}{2} e^{-s} ds \quad (20)$$

$$= \int_0^1 e^{-s} ds \quad (21)$$

$$= 1 - \frac{1}{e} \quad (22)$$

The Rayleigh's theorem is applied from (17) to (18)

2.

$$\int_{-\infty}^{\infty} t^2 \text{sinc}^4(t) dt = \int_{-\infty}^{\infty} (t \text{sinc}^2(t)) (t \text{sinc}^2(t)) dt \quad (23)$$

Using the derivative property, which states:

$$\mathcal{F}\{t^n g(t)\} = \left(\frac{j}{2\pi}\right)^n G^{[n]}(s), \quad (24)$$

where $G^{[n]}(s) = \frac{d^n G(s)}{ds^n}$ refers to the n^{th} order derivative of $G(s)$. We can deduce that $t \text{sinc}^2(t)$ transform to $\frac{j}{2\pi} \Lambda'(s)$. Here we take the derivative of $\Lambda(s)$ in a piecewise manner, which has the following graph:

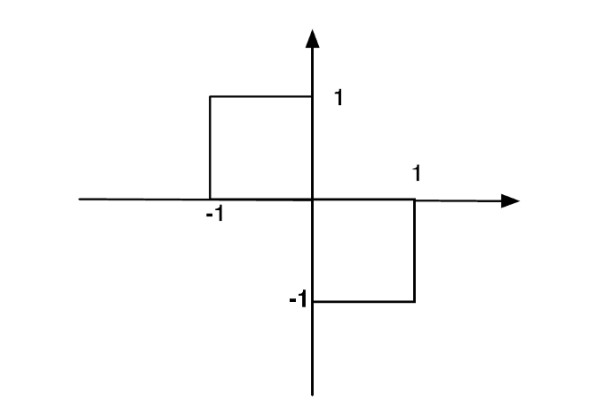


Figure 1: Graph of $\Lambda'(s)$

Hence, from the Rayleigh's theorem, we have

$$\int_{-\infty}^{\infty} t^2 \text{sinc}^4(t) dt = \int_{-\infty}^{\infty} \frac{j}{2\pi} \Lambda'(s) \left(\frac{j}{2\pi} \Lambda'(s) \right)^* ds \quad (25)$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \Lambda'(s) \Lambda'(s) ds \quad (26)$$

$$= \frac{1}{4\pi^2} (2) \quad (27)$$

$$= \frac{1}{2\pi^2} \quad (28)$$

Question 3

1. From the property of the Fourier Transform

$$\mathcal{F}\{f(t - \tau)\} = e^{-j2\pi\tau s} F(s) \quad (29)$$

$$\mathcal{F}\{g(t + \tau)\} = e^{j2\pi\tau s} G(s) \quad (30)$$

From the convolution theorem, the convolution becomes multiplication, which gives

$$\mathcal{F}\{f(t - \tau) * g(t + \tau)\} = e^{-j2\pi\tau s} F(s) e^{j2\pi\tau s} G(s) = F(s) G(s) = \mathcal{F}\{f(t) * g(t)\} = \mathcal{F}\{h(t)\}. \quad (31)$$

Therefore,

$$f(t - \tau) * g(t + \tau) = h(t) \quad (32)$$

2. Let $h(t) = f(t) * g(t)$, then

$$h(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau \quad (33)$$

since we know that $h(t) = \mathcal{F}^{-1}\{F(s)G(s)\}$ by the convolution theorem,

$$h(t) = \int_{-\infty}^{\infty} F(s) G(s) e^{j2\pi s t} ds \quad (34)$$

Hence

$$\int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau = \int_{-\infty}^{\infty} F(s) G(s) e^{j2\pi s t} ds \quad (35)$$

Evaluating this equality at $t = 0$, we can get

$$\int_{-\infty}^{\infty} f(\tau) g(-\tau) d\tau = \int_{-\infty}^{\infty} F(s) G(s) ds \quad (36)$$

(Replace τ with t gives the final expression).

Question 4

Define $U(s)$ and $F(s)$ as the Fourier transform of $u(t)$ and $f(t)$, respectively. Take the Fourier transform of both sides

$$(j2\pi s)^2 U(s) - U(s) = -F(s), \quad (37)$$

$$-4\pi^2 s^2 U(s) - U(s) = -F(s), \quad (38)$$

$$(1 + 4\pi^2 s^2) U(s) = F(s). \quad (39)$$

So we can express $U(s)$ as

$$U(s) = \frac{1}{1 + 4\pi^2 s^2} F(s). \quad (40)$$

We recognize $\frac{1}{1+4\pi^2 s^2}$ is the Fourier transform of $\frac{1}{2}e^{-|t|}$, that gives

$$u(s) = \mathcal{F} \left\{ \frac{1}{2} e^{-|t|} \right\} F(s). \quad (41)$$

According to the convolution theorem,

$$u(t) = \frac{1}{2} e^{-|t|} * f(t), \quad (42)$$

or, write out in full

$$u(t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|t-\tau|} f(\tau) d\tau \quad (43)$$

Question 5

1. Using the Fourier derivative theorem and the value at origin trick, we have the relationship between the n^{th} order moment and the n^{th} order derivative:

$$m_n = \int_{-\infty}^{\infty} t^n f(t) dt = \int_{-\infty}^{\infty} t^n f(t) e^{-j2\pi s t} dt \Big|_{s=0} \quad (44)$$

$$= \mathcal{F} \{ t^n f(t) \} \Big|_{s=0} \quad (45)$$

$$= \left(\frac{j}{2\pi} \right)^n \frac{d^n F(s)}{ds} \Big|_{s=0} \quad (46)$$

$$= \left(\frac{j}{2\pi} \right)^n F^{[n]}(0) \quad (47)$$

2. Consider the mathematical format of Taylor series of $F(s)$ around 0:

$$F(s) = \sum_{n=0}^{\infty} F^{[n]}(0) \frac{s^n}{n!}. \quad (48)$$

In Q1, we have proved that $m_n = \left(\frac{j}{2\pi} \right)^n F^{[n]}(0)$. It can be directly substituted into Eq.(48) to replace the n^{th} order derivative term in Taylor series, i.e.,

$$F(s) = \sum_{n=0}^{\infty} F^{[n]}(0) \frac{s^n}{n!}, \quad (49)$$

$$= \sum_{n=0}^{\infty} (-j2\pi)^n \left(\frac{j}{2\pi} \right)^n F^{[n]}(0) \frac{s^n}{n!}, \quad (50)$$

$$= \sum_{n=0}^{\infty} (-j2\pi)^n m_n \frac{s^n}{n!}. \quad (51)$$

Q.E.D.

Question 6

1. First, we determine the first derivative of the piece-wise function as:

$$\frac{df(t)}{dt} = \begin{cases} 0 & \text{when } t < 0 \\ \frac{a_1}{t_1} = a'_1 & \text{when } 0 < t \leq t_1 \\ \frac{a_2 - a_1}{t_2 - t_1} = a'_2 & \text{when } t_1 < t \leq t_2 \\ \frac{a_3 - a_2}{t_3 - t_2} = a'_3 & \text{when } t_2 < t \leq t_3 \\ \frac{a_4 - a_3}{t_4 - t_3} = a'_4 & \text{when } t_3 < t \leq t_4 \\ 0 & \text{when } t > t_4 \end{cases},$$

Therefore, it is obvious that the second derivative of $f(t)$ is a series of delta functions with different amplitudes according to the property of the delta function:

$$\frac{d^2f(t)}{dt^2} = a'_1\delta(t) + (a'_2 - a'_1)\delta(t - t_1) + (a'_3 - a'_2)\delta(t - t_2) + (a'_4 - a'_3)\delta(t - t_3) - a'_4\delta(t - t_4) \quad (52)$$

2. Using the Fourier derivative theorem, i.e.,

$$\frac{d^2f(t)}{dt^2} \supset (j2\pi s)^2 F(s), \quad (53)$$

where $f(t) \supset F(s)$. Therefore, to determine the $F(s)$, we need first calculate the Fourier transform of the second derivative. According to the Fourier shifting theorem, it is easy to find:

$$\frac{d^2f(t)}{dt^2} \supset a'_1 + (a'_2 - a'_1)e^{-j2\pi t_1 s} + (a'_3 - a'_2)e^{-j2\pi t_2 s} + (a'_4 - a'_3)e^{-j2\pi t_3 s} - a'_4e^{-j2\pi t_4 s} \quad (54)$$

$$= (j2\pi s)^2 F(s) \quad (55)$$

Finally, rearranging both sides of the equation, it can be simply obtained:

$$F(s) = \frac{a'_1 + (a'_2 - a'_1)e^{-j2\pi t_1 s} + (a'_3 - a'_2)e^{-j2\pi t_2 s} + (a'_4 - a'_3)e^{-j2\pi t_3 s} - a'_4e^{-j2\pi t_4 s}}{(j2\pi s)^2}, \quad (56)$$

$$\begin{aligned} &= -\frac{1}{4\pi^2 s^2} \left[\left(\frac{a_1}{t_1} \right) + \left(\frac{a_2 - a_1}{t_2 - t_1} - \frac{a_1}{t_1} \right) e^{-j2\pi t_1 s} + \left(\frac{a_3 - a_2}{t_3 - t_2} - \frac{a_2 - a_1}{t_2 - t_1} \right) e^{-j2\pi t_2 s} \right. \\ &\quad \left. + \left(\frac{a_4 - a_3}{t_4 - t_3} - \frac{a_3 - a_2}{t_3 - t_2} \right) e^{-j2\pi t_3 s} - \left(\frac{a_4 - a_3}{t_4 - t_3} \right) e^{-j2\pi t_4 s} \right] \end{aligned} \quad (57)$$

Question 7

1. The signal is made out of the Heaviside step function $H(t)$, and we know its Fourier transformation to be:

$$\mathcal{F}\{H(t)\} = \frac{1}{2}\delta(s) - \frac{j}{2\pi s}. \quad (58)$$

The signal can be written as:

$$f(t) = H(t - 1) - H(-(t + 1)). \quad (59)$$

Therefore, we can determine the Fourier transform of the signal in Filter 1 as:

$$\mathcal{F}\{f(t)\} = \mathcal{F}\{H(t - 1)\} - \mathcal{F}\{H(-(t + 1))\} \quad (60)$$

$$= e^{-2\pi j s} \mathcal{F}\{H(t)\}(s) - e^{2\pi j s} \mathcal{F}\{H(t)\}(-s) \quad (61)$$

$$= e^{-2\pi j s} \mathcal{F}\{H(t)\}(s) - e^{2\pi j s} \mathcal{F}\{H(t)\}(-s) \quad (62)$$

$$= e^{-2\pi j s} \left(\frac{1}{2}\delta(s) - \frac{j}{2\pi s} \right) - e^{2\pi j s} \left(\frac{1}{2}\delta(-s) - \frac{j}{2\pi(-s)} \right) \quad (63)$$

$$= \frac{1}{2} (e^{-2\pi j s} - e^{2\pi j s}) \delta(s) - \frac{j}{2\pi s} (e^{-2\pi j s} + e^{2\pi j s}) \quad (64)$$

$$= -j(\sin 2\pi s)\delta(s) - \frac{j}{\pi s} \cos 2\pi s \quad (65)$$

$$= -\frac{j}{\pi s} \cos 2\pi s. \quad (66)$$

$$(67)$$

Note: From Eq.(65) $\rightarrow$ Eq.(66), consider the property of a delta function, we have the first term $-j(\sin 2\pi s)\delta(s) = 0$.

2. Filter 2 can be described as:

$$\sin(2\pi|t|) = \sin(2\pi t)\text{sgn}(t). \quad (68)$$

Taking the Fourier transform we get

$$\mathcal{F}\{\sin(2\pi t)\text{sgn}(t)\} = \mathcal{F}\{\sin(2\pi t)\} * \mathcal{F}\{\text{sgn}(t)\}. \quad (69)$$

We know the Fourier transform of a signum function is

$$\mathcal{F}\{\text{sgn}(t)\} = \mathcal{F}\{2H(t) - 1\} = \frac{1}{j\pi s}, \quad (70)$$

therefore, substitute it into Eq.(69) we get:

$$\mathcal{F}\{\sin(2\pi|t|)\} = \frac{1}{2j}(\delta(s-1) - \delta(s+1)) * \frac{1}{\pi j s} \quad (71)$$

$$= \frac{-1}{2\pi(s-1)} + \frac{1}{2\pi(s+1)} \quad (72)$$

$$= \frac{-s-1+s-1}{2\pi(s-1)(s+1)}. \quad (73)$$

After simplification, the final answer is

$$\mathcal{F}\{\sin(2\pi|t|)\} = \frac{1}{\pi(1-s^2)} \quad (74)$$