

HomeworkAns 1

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Question 1

The function in the figure can be expressed as the sum of two scaled and shifted Λ functions, which has the following form:

$$2\Lambda\left(\frac{t-2}{2}\right) + 2.5\Lambda\left(\frac{t-4}{2}\right). \quad (1)$$

Thus, using the shift and stretch theorems, we find the following Fourier transform

$$2\Lambda\left(\frac{t-2}{2}\right) + 2.5\Lambda\left(\frac{t-4}{2}\right) \supset 2e^{-2j\pi s(2)} 2\text{sinc}^2(2s) + 2.5e^{-2j\pi s(4)} 2\text{sinc}^2(2s), \quad (2)$$

$$\supset \text{sinc}^2(2s) (4e^{-4j\pi s} + 5e^{-8j\pi s}). \quad (3)$$

Question 2

Since the Fourier transform of $\cos(2\pi at)$ is $\frac{1}{2}(\delta(s-a) + \delta(s+a))$, the output of the filter is

$$\mathcal{F}^{-1} \left\{ \frac{1}{2} (\delta(s-a) + \delta(s+a)) H(s) \right\} = \mathcal{F}^{-1} \left\{ \frac{1}{2} (\delta(s-a)H(a) + \delta(s+a)H(-a)) \right\} \quad (4)$$

$$= \mathcal{F}^{-1} \left\{ \frac{1}{2} (\delta(s-a) + \delta(s+a)) H(a) \right\} \quad (\text{because } H \text{ is even}) \quad (5)$$

$$= \cos(2\pi at)H(a). \quad (6)$$

Question 3

From the hint, we have

$$\text{Re}\{F(s)\} = \frac{F(s) + F^*(s)}{2}, \quad (7)$$

and since $f(t)$ is real, we have the symmetry of its Fourier transform

$$F^*(s) = F(-s). \quad (8)$$

Thus

$$\text{Re}\{F(s)\} = \frac{F(s) + F(-s)}{2}. \quad (9)$$

Combined with the third fact, we have

$$|t|e^{-|t|} = \mathcal{F}^{-1} \{ \text{Re}\{F(s)\} \} \quad (10)$$

$$= \mathcal{F}^{-1} \left\{ \frac{F(s) + F(-s)}{2} \right\} \quad (11)$$

$$= \frac{1}{2} (\mathcal{F}^{-1} \{F(s)\} + \mathcal{F}^{-1} \{F(-s)\}) \quad (12)$$

$$= \frac{1}{2} (f(t) + f(-t)). \quad (13)$$

Now, considering $t > 0$ with the second fact,

$$te^{-t} = |t|e^{-|t|} = \frac{1}{2}(f(t) + f(-t)) = \frac{1}{2}f(t). \quad (14)$$

Now, we can say

$$f(t) = \begin{cases} 2te^{-t} & t > 0 \\ 0 & t \leq 0 \end{cases} \quad (15)$$

Question 4

1. From the figure, we can conclude that $f(t)$ is real and decomposed as

$$f(t) = f_o(t) + f_e(t), \quad (16)$$

where $f_o(t) = \frac{1}{2}[f(t) - f(-t)]$ and $f_e(t) = \frac{1}{2}[f(t) + f(-t)]$ denote the odd and even part of $f(t)$, respectively.

Since $f_e(t)$ gives rise to a transform that is real and even while $f_o(t)$ results in one that is purely imaginary and odd, the real part of $F(s)$ will be the Fourier transform of $f_e(t)$. We draw the figure of $f_e(t)$ as Figure 1, which is effectively $\frac{1}{2}\Lambda(t)$. The Fourier transform of $\frac{1}{2}\Lambda(t)$, hence the real part of $F(s)$, will be $\frac{1}{2}\text{sinc}^2(s)$.

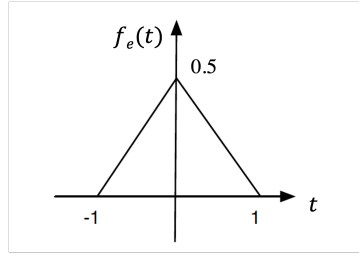


Figure 1: Graph of the even part of $f(t)$

2. We know that

$$g(t) = f(t) - f(-t) = 2f_o(t). \quad (17)$$

The Fourier transform of $f_o(t)$ is j times the imaginary part of $F(s)$. It follows that

$$G(s) = \mathcal{F}\{2f_o(t)\} = 2j \frac{\sin(2\pi s) - 2\pi s}{4\pi^2 s^2} = j \frac{\sin(2\pi s) - 2\pi s}{2\pi^2 s^2} \quad (18)$$

Question 5

1. $f_1(t)$ is a scaled triangle function of unit area, i.e., $\frac{1}{N}\Lambda(t/N)$. Therefore, its Fourier transform can be written as:

$$\mathcal{F}\{f_1(t)\} = \mathcal{F}\left\{\frac{1}{N}\Lambda(t/N)\right\}, \quad (19)$$

$$= \mathcal{F}\left\{\frac{1}{N}\Pi(t/N) * \frac{1}{N}\Pi(t/N)\right\}, \quad (20)$$

$$= \mathcal{F}\left\{\frac{1}{N}\Pi(t/N)\right\} \mathcal{F}\left\{\frac{1}{N}\Pi(t/N)\right\}, \quad (21)$$

$$= \text{sinc}^2(Ns). \quad (22)$$

2. By directly integrating $f_2(t)$, we have:

$$\mathcal{F}\{f_2(t)\} = \int_{-\infty}^{\infty} \frac{1}{N} e^{-\frac{t}{N}} H(t) e^{-j2\pi s t} dt, \quad (23)$$

$$= \frac{1}{N} \int_0^{\infty} e^{-t(\frac{1}{N} + j2\pi s)} dt, \quad (24)$$

$$= -\frac{e^{-t(\frac{1}{N} + j2\pi s)}}{1 + j2\pi s N} \Big|_{t=0}^{t=\infty} = \frac{1}{1 + j2\pi s N}. \quad (25)$$

3. $f_3(t)$ is just the horizontal coordinate flipping version of $f_2(t)$:

$$\mathcal{F}\{f_3(t)\} = \int_{-\infty}^{\infty} \frac{1}{N} e^{\frac{t}{N}} H(-t) e^{-j2\pi s t} dt, \quad (26)$$

$$= \frac{1}{N} \int_{-\infty}^0 e^{t(\frac{1}{N} - j2\pi s)} dt, \quad (27)$$

$$= \frac{e^{t(\frac{1}{N} - j2\pi s)}}{1 - j2\pi s N} \Big|_{t=-\infty}^{t=0} = \frac{1}{1 - j2\pi s N}. \quad (28)$$

4. Through observation, we can find the relationship of $f_4(t) = (f_2(t) + f_3(t))/2$. Therefore, it is easy to derive its Fourier transform with:

$$\mathcal{F}\{f_4(t)\} = (\mathcal{F}\{f_2(t)\} + \mathcal{F}\{f_3(t)\})/2, \quad (29)$$

$$= \left(\frac{1}{1 + j2\pi s N} + \frac{1}{1 - j2\pi s N} \right) / 2 = \frac{1}{1 + (2\pi s N)^2}. \quad (30)$$

5. When $N \rightarrow 0$, it is easy to show that the Fourier transforms of the $f_1(t)$ and $f_4(t)$ approach 1. $f_1(t)$ and $f_4(t)$ see their width become null and their amplitudes become infinite while maintaining an area included under the curve constant and equal to 1. It can therefore be said that the $f_1(t)$ and $f_4(t)$ approach the delta function and their Fourier transforms approach the transform of the delta function.

Question 6

1. In the polar coordinate system, the signal $f(r)$ that characterizes an ideal ring with a radius of r_0 should be:

$$f(r) = \delta(r - r_0). \quad (31)$$

2. Considering the Fourier-Bessel transform in the course notes, we have the following derivation:

$$\mathcal{B}\{f(r)\} = 2\pi \int_0^{\infty} r f(r) J_0(2\pi r \rho) dr, \quad (32)$$

$$= 2\pi \int_0^{\infty} r \delta(r - r_0) J_0(2\pi r \rho) dr, \quad (33)$$

$$= 2\pi r_0 J_0(2\pi r_0 \rho). \quad (34)$$

Note: the convolution theorem of a Dirac impulse function is used in the derivation of (33) $\rightarrow$ (34).

Add: this question doesn't mention a unit strength, so both unit/ scaled versions would be correct.

Question 7

1. In the 2D regular coordinate system, the signal $f(x, y)$ that characterises this mask should be:

$$f(x, y) = \Pi\left(\frac{x}{a}, \frac{y}{a}\right) * \frac{1}{XY} \mathfrak{W}\left(\frac{x}{X}, \frac{y}{Y}\right). \quad (35)$$

2. Considering the 2D Fourier transform in the handout, we have the following derivation:

$$\mathcal{F}\{f(x, y)\} = \mathcal{F}\left\{\Pi\left(\frac{x}{a}, \frac{y}{a}\right) * \frac{1}{XY} \mathbf{\Pi}\left(\frac{x}{X}, \frac{y}{Y}\right)\right\}, \quad (36)$$

$$= \mathcal{F}\left\{\Pi\left(\frac{x}{a}, \frac{y}{a}\right)\right\} \mathcal{F}\left\{\frac{1}{XY} \mathbf{\Pi}\left(\frac{x}{X}, \frac{y}{Y}\right)\right\}, \quad (37)$$

$$= a^2 \text{sinc}(u, v) \mathbf{\Pi}(Xu, Yv). \quad (38)$$

Note: Fourier convolution theorem is used in the derivation of (36) $\rightarrow$ (37).