Fourier transform theorems

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ELEC8501: The Fourier Transform and Its Applications
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https://www.eee.hku.hk/~elec8501

1 General properties of Fourier transform

2 General properties of Fourier transform in 2D

3 Relationships between the two domains

Fourier transform properties

The general question: When we "do something" to the function (signal) in the time domain, what is the effect in the frequency domain?

Need to be able to prove the properties. You can often go back to the integral, or make use of other known properties.

More importantly, you should be able to use the properties (i.e., make arguments with these properties)

(1) addition

Addition (linearity):

$$\boxed{a \cdot g(t) + b \cdot h(t) \supset a \cdot G(s) + b \cdot H(s)} \tag{1}$$

(2) similarity

Similarity:

$$\boxed{g(at) \supset \frac{1}{|a|} G\left(\frac{s}{a}\right)} \tag{2}$$

Compression in time $\rightarrow$ expansion in frequency

(3) shift

Shift:

$$\boxed{g(t-a) \supset e^{-j2\pi as} G(s)} \tag{3}$$

How is cosine and sine related? And in between?
Imagine you are translating $\Pi(t)$ to $\Pi(t-\frac{1}{2})$. What is the evolution in the frequency domain?

shift

Exercise: Find $\mathcal{F}\left\{\frac{\cos \pi t}{\pi(t-\frac{1}{2})}\right\}$.

shift

What do we know about the Fourier transform of a periodic function, where $g(t+T) = g(t)$?

(4) modulation

Modulation:

$$\boxed{g(t)\cos(2\pi at) \supset \frac{1}{2}\left[G(s-a)+G(s+a)\right]} \tag{4}$$

(5) convolution

Convolution:

$$\boxed{g(t) * h(t) \supset G(s)H(s)} \tag{5}$$

Note: $H(s)$ is often called the *transfer function*: specifying the **weight** to amplify/attenuate a particular frequency content

(6) duality

Because of the similarity between the forward and inverse Fourier transform, we can have the following duality theorems:

$$\boxed{\begin{aligned} g(t-a) &\supset e^{-j2\pi as}G(s) \\ e^{-j2\pi at}g(t) &\supset G(s+a) \end{aligned}} \tag{6}$$

$$\boxed{\begin{aligned} g(t) * h(t) &\supset G(s)H(s) \\ g(t)h(t) &\supset G(s) * H(s) \end{aligned}} \tag{7}$$

(7) autocorrelation

When a function convolves with itself, we have

$$f(t) * f(t) = \int_{-\infty}^{\infty} f(u)f(t-u)du \tag{8}$$

Suppose we do not "flip" the function inside the integration; we write

$$f(t) \star f(t) = \int_{-\infty}^{\infty} f(u)f(u-t)du \tag{9}$$

$$= \int_{-\infty}^{\infty} f(w)f(w+t)dw \tag{10}$$

(with $w = u - t$)

autocorrelation

Some important properties:

- If $f(t)$ is even, $f(t) \star f(t) = f(t) * f(t)$
- $f(t) \star f(t)$ is an even function
- It is maximum at the origin, i.e.,

$$\int_{-\infty}^{\infty} f(u)f(u+t)du \leq \int_{-\infty}^{\infty} [f(u)]^2 du \tag{11}$$

autocorrelation

In general, $f(t)$ is complex, and we define the autocorrelation to be $f^* \star f$:

$$f^*(t) \star f(t) = \int_{-\infty}^{\infty} f(u)f^*(u-t)du \tag{12}$$

$$= \int_{-\infty}^{\infty} f^*(w)f(w+t)dw \tag{13}$$

(with $w = u - t$)

autocorrelation

Sometimes, we want to normalize the autocorrelation:

$$\gamma(t) = \frac{\int_{-\infty}^{\infty} f^*(u)f(u+t)du}{\int_{-\infty}^{\infty} f^*(u)f(u)du} \tag{14}$$

Because the maximum is at the origin, we have:

- $\gamma(0) = 1$
- $\gamma(t) \leq 1$
- it is an even function

autocorrelation

Autocorrelation:

$$f^*(t) \star f(t) \supset |F(s)|^2 \tag{15}$$

(8) cross-correlation

We can generalize autocorrelation to cross-correlation:

$$g^*(t) \star h(t) = \int_{-\infty}^{\infty} h(u) g^*(u-t) du \tag{16}$$

$$= \int_{-\infty}^{\infty} g^*(w) h(w+t) dw \tag{17}$$

(with $w = u - t$)
We think of this as "g^* scans h", which is applicable in pattern recognition

cross-correlation

Cross-correlation:

$$g^*(t) \star h(t) \supset G^*(s) H(s) \tag{18}$$

(9) derivative

Derivative:

$$\frac{d}{dt} g(t) \supset j2\pi s G(s) \tag{19}$$

derivative

We can have higher order derivatives; we can also have the dual to this theorem.

$$\frac{d^2}{dt^2} g(t) \supset (j2\pi s)^2 G(s) \tag{20}$$

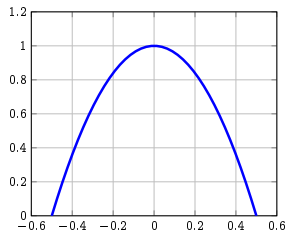
$$t g(t) \supset \frac{j}{2\pi} \frac{d}{ds} G(s) \tag{21}$$

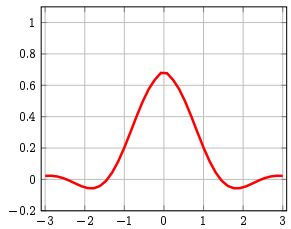
derivative

Exercise: Find $\mathcal{F}\left\{te^{-\pi t^2}\right\}$, and $\mathcal{F}\left\{(1-4t^2)\Pi(t)\right\}$.

derivative

Plots of $(1-4t^2)\Pi(t)$ and its Fourier transform:





(10) derivative of convolution

When $h(t) = f(t) * g(t)$, we have

$$\frac{d}{dt}h(t) = \frac{d}{dt}f(t) * g(t) \tag{22}$$

$$= f(t) * \frac{d}{dt}g(t) \tag{23}$$

Therefore,

$$\frac{d}{dt}f(t) * g(t) = j2\pi s F(s)G(s) \tag{24}$$

$$f(t) * \frac{d}{dt}g(t) = j2\pi s F(s)G(s) \tag{25}$$

summary

Theorem	time	frequency
addition	$g(t)+h(t)$	$G(s)+H(s)$
similarity	$g(at)$	$\frac{1}{\lvert a\rvert}G\left(\frac{s}{a}\right)$
shift	$g(t-a)$	$e^{-j2\pi as}G(s)$
modulation	$g(t)\cos(2\pi at)$	$\frac{1}{2}\left[G(s-a)+G(s+a)\right]$
convolution	$g(t)*h(t)$	$G(s)H(s)$
product	$g(t)h(t)$	$G(s)*H(s)$
autocorrelation	$f^*(t)\star f(t)$	$\lvert F(s)\rvert^2$
cross-correlation	$g^*(t)\star h(t)$	$G^*(s)H(s)$
derivative	$\frac{d}{dt}g(t)$	$j2\pi sG(s)$
derivative of convolution	$\frac{d}{dt}f(t)*g(t)$	$j2\pi sF(s)G(s)$

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general properties

For 2D functions, many properties relating "time" and "frequency" in 1D can be generalized to similar relationships between "space" and "spatial frequency"

Example: translation, stretching, modulation, convolution, correlation, ...

But what about rotation? If we rotate a 2D function (clockwise by an angle θ), what is the effect in the (spatial) frequency domain?

rotation

$$g(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) \supset G(u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta) \tag{26}$$

$$g(r, \theta - k) \supset G(\rho, \phi - k) \tag{27}$$

The expression in Cartesian coordinates looks complicated; using polar coordinates is simpler. The conclusion is the same: The Fourier domain experiences the same rotation (same direction, same angle).

rotation

Proof:

$$\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix}x\\ y\end{bmatrix} \tag{28}$$

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} = \begin{bmatrix}\hat{x}\cos\theta - \hat{y}\sin\theta\\ \hat{x}\sin\theta + \hat{y}\cos\theta\end{bmatrix} \tag{29}$$

$$d\hat{x}\, d\hat{y} = \begin{vmatrix}\frac{\partial\hat{x}}{\partial x} & \frac{\partial\hat{y}}{\partial x}\\ \frac{\partial\hat{x}}{\partial y} & \frac{\partial\hat{y}}{\partial y}\end{vmatrix} dx\, dy = dx\, dy \tag{30}$$

rotation

$$
\begin{aligned}
g(\hat{x},\hat{y}) &\supset \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\hat{x},\hat{y}) e^{-j2\pi(xu+yv)}\, dx\, dy && (31)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\hat{x},\hat{y}) e^{-j2\pi(u\hat{x}\cos\theta - u\hat{y}\sin\theta + v\hat{x}\sin\theta + v\hat{y}\cos\theta)} d\hat{x}\, d\hat{y} && (32)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\hat{x},\hat{y}) e^{-j2\pi[\hat{x}(u\cos\theta + v\sin\theta) + \hat{y}(v\cos\theta - u\sin\theta)]} d\hat{x}\, d\hat{y} && (33)\\
&= G(u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta) && (34)
\end{aligned}
$$

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Rayleigh's theorem

Rayleigh:

$$\boxed{\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(s)|^2 ds} \qquad (35)$$

- This is true for the Fourier transform formula we are using; otherwise, there may be extra factors of 2π
- Interpretation: These compute the "energy" of the function
- Other names: Parseval's theorem, Plancherel's theorem

Rayleigh's theorem

Proof:

$$
\begin{aligned}
\int_{-\infty}^{\infty} |g(t)|^2 dt &= \int_{-\infty}^{\infty} g(t) g^*(t) e^{-j2\pi\hat{s}t} dt \Big|_{\hat{s}=0}\\
&= \mathcal{F}\{g(t)g^*(t)\} \Big|_{\hat{s}=0}\\
&= \mathcal{F}\{g(t)\} * \mathcal{F}\{g^*(t)\} \Big|_{\hat{s}=0}\\
&= G(\hat{s}) * G^*(-\hat{s}) \Big|_{\hat{s}=0}\\
&= \int_{-\infty}^{\infty} G(s) G^*(s-\hat{s}) ds \Big|_{\hat{s}=0}\\
&= \int_{-\infty}^{\infty} |G(s)|^2 ds
\end{aligned}
$$

Power theorem

Generalizing Rayleigh's theorem:

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(s)G^*(s)ds \tag{36}$$

Exercises: Evaluate the following:

- $\int_{-\infty}^{\infty} \text{sinc}(t)dt$
- $\int_{-\infty}^{\infty} \text{sinc}^2(t)dt$
- $\int_{-\infty}^{\infty} \text{sinc}^4(t)dt$

definite integral

Definite integral = value at origin:

$$\int_{-\infty}^{\infty} g(t)dt = G(0) \tag{37}$$

moments

First moment:

$$\int_{-\infty}^{\infty} tg(t)dt = \frac{j}{2\pi}G'(0) \tag{38}$$

Second moment (moment of inertia):

$$\int_{-\infty}^{\infty} t^2 g(t)dt = -\frac{1}{4\pi^2}G''(0) \tag{39}$$

centroid

Centroid:

$$\langle t \rangle_g = \frac{\int_{-\infty}^{\infty} tg(t)dt}{\int_{-\infty}^{\infty} g(t)dt} = \frac{j}{2\pi}\frac{G'(0)}{G(0)} \tag{40}$$

Mean-square abscissa:

$$\langle t^2 \rangle_g = \frac{\int_{-\infty}^{\infty} t^2 g(t)dt}{\int_{-\infty}^{\infty} g(t)dt} = -\frac{1}{4\pi^2}\frac{G''(0)}{G(0)} \tag{41}$$

variance

Variance:

$$\sigma_g^2 = \langle (t - \langle t \rangle)^2 \rangle = -\frac{1}{4\pi^2}\frac{G''(0)}{G(0)} + \left[\frac{j}{2\pi}\frac{G'(0)}{G(0)}\right]^2 \tag{42}$$

variance

We would normally expect that for $g(t) = f(t) * h(t)$, that $g(t)$ is "wider" than the other two. In fact, we can quantify this statement:

$$\sigma_g^2 = \sigma_f^2 + \sigma_h^2 \tag{43}$$

equivalent width

In fact, we often want some ways to quantify the "width" of a function. One way is called the *equivalent width*:

$$\text{equivalent width} = \frac{\int_{-\infty}^{\infty} f(t)dt}{f(0)} \tag{44}$$

The product of the equivalent widths in the two domain is unity:

$$\frac{\int_{-\infty}^{\infty} f(t)dt}{f(0)} \cdot \frac{\int_{-\infty}^{\infty} F(s)ds}{F(0)} = 1 \tag{45}$$

mean-square width

The variance can also serve as the width measurement. To allow it to measure the width of an oscillatory signal (where the area is zero), we consider the variance of the energy distribution,

$$(\Delta t)^2 = \sigma_{|g|^2}^2 = \frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt} - \left[\frac{\int_{-\infty}^{\infty} t|g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt}\right]^2 \tag{46}$$

And here is a famous uncertainty relation:

$$\Delta t \Delta s \geq \frac{1}{4\pi} \tag{47}$$

There is a limit to how narrow we can make the function is both time and frequency domains!