

Useful functions and their transforms (II)

Dr. Edmund Lam

Department of Electrical and Electronic Engineering
The University of Hong Kong

ELEC8501: The Fourier Transform and Its Applications
(First Semester, 2022-23)

<https://www.eee.hku.hk/~elec8501>

- 1 Some useful 2D functions
- 2 Fourier transform of some useful 2D functions
- 3 The delta function in 2D
- 4 Fourier transform related to the delta function

2D functions

We now consider 2D function $f(x, y)$

time $\rightarrow$ space (spatial domain)

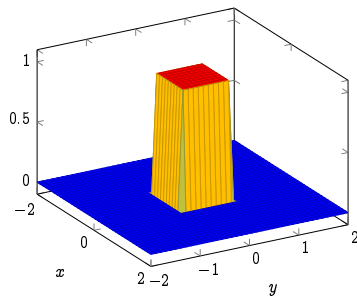
visualization is harder...

2D rect function

We define a 2D rect function as

$${}^2\Pi(x, y) = \begin{cases} 1 & |x| < \frac{1}{2} \text{ and } |y| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

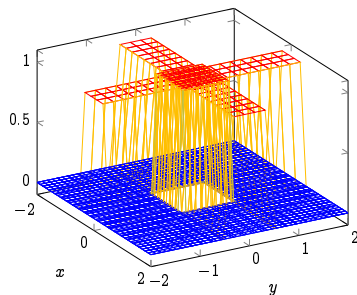
(The superscript “2” is often omitted when there’s no ambiguity)



separable function

An important characteristic is that this is a *separable function* (in Cartesian coordinates), where we can express as $f(x, y) = f_X(x)f_Y(y)$.

$$^2\Pi(x, y) = \Pi(x)\Pi(y) \quad (2)$$



polar coordinates

In 2D, we can express functions not only in Cartesian coordinates, but also in polar coordinates.

Separable in Cartesian coordinates:

$$f(x, y) = f_X(x)f_Y(y) \quad (3)$$

Separable in polar coordinates:

$$f(r, \theta) = f_R(r)f_\Theta(\theta) \quad (4)$$

Furthermore, a function is *circularly symmetric* if:

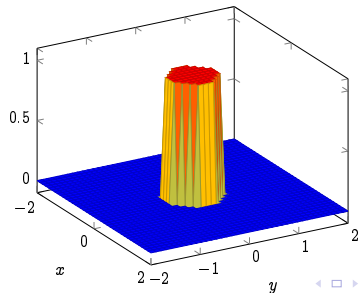
$$f(r, \theta) = f_R(r) \quad (5)$$

circ function

We define the *circ function* as

$$\Pi(r) = \begin{cases} 1 & |r| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Usually, it is the *context* that tells us it's a 2D function in polar coordinates, rather than a 1D function!

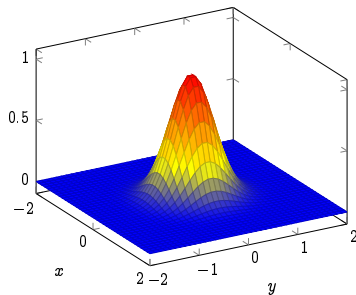


2D Gaussian

A special case: the 2D Gaussian function is separable in both Cartesian and polar coordinates, actually circularly symmetric

$$f(x, y) = e^{-\pi(x^2+y^2)} \quad (7)$$

$$f(r, \theta) = e^{-\pi r^2} \quad (8)$$



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2D Fourier transform

Fourier transform of a 2D function $f(x, y)$ is another 2D function $F(u, v)$

frequency $\rightarrow$ spatial frequency

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(xu+yv)} dx dy \quad (9)$$

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(xu+yv)} du dv \quad (10)$$

We can write ${}^2\mathcal{F}\{f(x, y)\} = F(u, v)$, but usually the superscript “2” is omitted when there’s no ambiguity

n-D Fourier transform

If we write $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$, then $\mathbf{x} \cdot \mathbf{u} = xu + yv$, and therefore

$$F(\mathbf{u}) = \int_{-\infty}^{\infty} f(\mathbf{x}) e^{-j2\pi(\mathbf{x} \cdot \mathbf{u})} d\mathbf{x} \quad (11)$$

$$f(\mathbf{x}) = \int_{-\infty}^{\infty} F(\mathbf{u}) e^{j2\pi(\mathbf{u} \cdot \mathbf{x})} d\mathbf{u} \quad (12)$$

We can extend this to higher dimensions.

Separable functions

2D Fourier transform is much simpler if the function is separable, i.e.,
 $f(x, y) = f_X(x)f_Y(y)$.

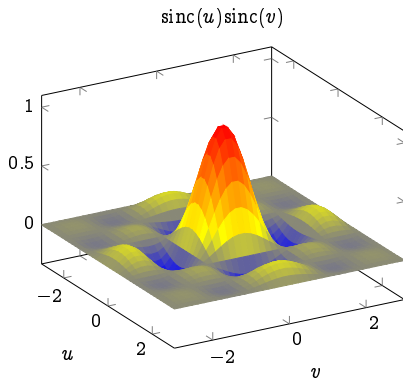
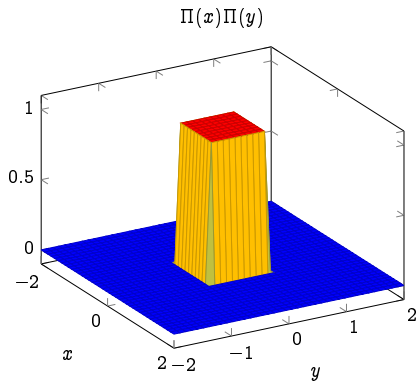
$$\mathcal{F}\{f(x, y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(xu+yv)} dx dy \quad (13)$$

$$= \left(\int_{-\infty}^{\infty} f_X(x) e^{-j2\pi ux} dx \right) \left(\int_{-\infty}^{\infty} f_Y(y) e^{-j2\pi vy} dy \right) \quad (14)$$

$$F(u, v) = F_U(u)F_V(v) \quad (15)$$

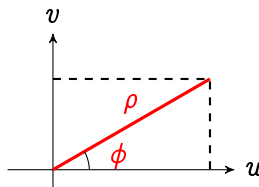
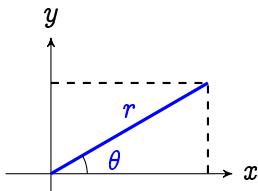
(Note: This is NOT convolution!)

An example



separable in polar coordinates

We should do coordinate transformation when the function is separable in polar coordinates.



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\tan \theta = \frac{y}{x}$$

$$u = \rho \cos \phi \quad (16)$$

$$v = \rho \sin \phi \quad (17)$$

$$\rho = \sqrt{u^2 + v^2} \quad (18)$$

$$\tan \phi = \frac{v}{u} \quad (19)$$

separable in polar coordinates

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(xu+yv)} dx dy \quad (20)$$

$$F(\rho, \phi) = \int_0^{2\pi} \int_0^{\infty} f_R(r) f_{\Theta}(\theta) e^{-j2\pi(r \cos \theta \cdot \rho \cos \phi + r \sin \theta \cdot \rho \sin \phi)} r dr d\theta \quad (21)$$

$$= \int_0^{2\pi} \int_0^{\infty} f_R(r) f_{\Theta}(\theta) e^{-j2\pi r \rho \cos(\theta - \phi)} r dr d\theta \quad (22)$$

For circularly symmetric functions: $f_{\Theta}(\theta)$ is a constant (e.g. = 1)

$$F_{\rho}(\rho) = \int_0^{\infty} r f_R(r) \left(\int_0^{2\pi} e^{-j2\pi r \rho \cos(\theta - \phi)} d\theta \right) dr \quad (23)$$

Bessel functions

In dealing with circular objects, we often find the Bessel functions at work! (cf. sinusoidal for 1D)

$$J_0(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-jr \cos \theta} d\theta \quad (24)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(r \cos \theta) d\theta \quad (25)$$

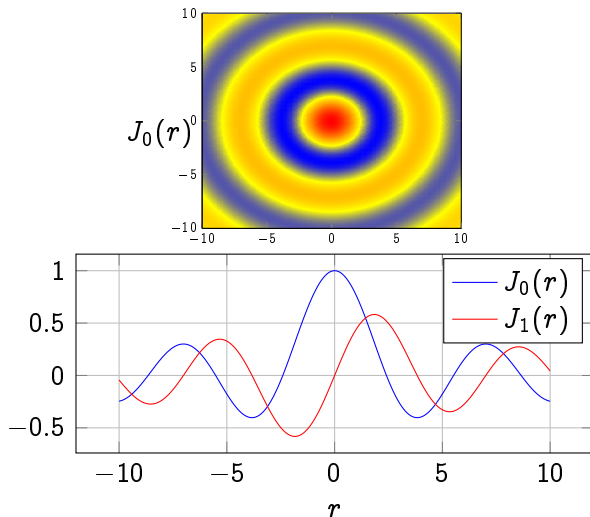
J means Bessel function of the “first kind”; 0 means “order zero”.

Also relevant for us is the Bessel function of the first kind, order one:

$$J_1(r) = \frac{1}{r} \int_0^r u J_0(u) du \quad (26)$$

Bessel functions

What they look like:



Fourier-Bessel transform

Putting the Bessel function in Eq. (23), and with similar derivations for the inverse Fourier transform, we have:

$$F_{\rho}(\rho) = 2\pi \int_0^{\infty} r f_R(r) J_0(2\pi r \rho) dr \quad (27)$$

$$f_R(r) = 2\pi \int_0^{\infty} \rho F_{\rho}(\rho) J_0(2\pi r \rho) d\rho \quad (28)$$

- 2D Fourier transform of a circularly symmetric function is also circularly symmetric
- This is also known as the Fourier-Bessel transform, or Hankel transform
- We may use $\mathcal{B}\{\cdot\}$ to denote the transform, e.g.,

$$\mathcal{B}\{f_R(r)\} = F_{\rho}(\rho) \quad (29)$$

Fourier-Bessel transform of circ

We want to compute the 2D Fourier transform of the circ function $\Pi(r)$:

$$\mathcal{B}\{\Pi(r)\} = 2\pi \int_0^{1/2} r(1) J_0(2\pi r \rho) dr \quad (30)$$

Let $\hat{r} = 2\pi r \rho$, so $r = \frac{\hat{r}}{2\pi \rho}$ and $dr = \frac{d\hat{r}}{2\pi \rho}$

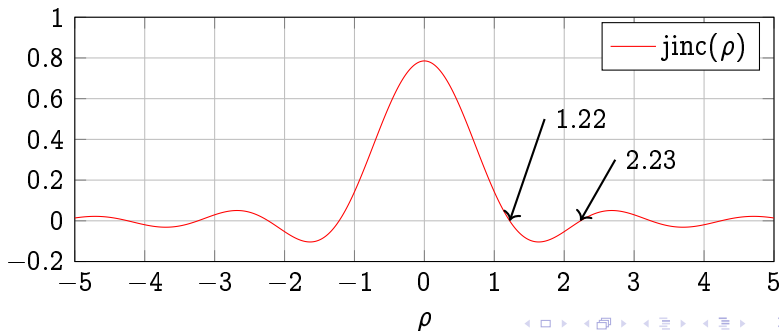
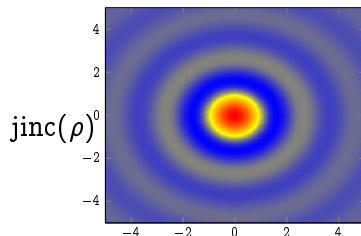
$$\mathcal{B}\{\Pi(r)\} = \frac{2\pi}{(2\pi \rho)^2} \underbrace{\int_0^{\pi \rho} \hat{r} J_0(\hat{r}) d\hat{r}}_{\pi \rho J_1(\pi \rho)} \quad (31)$$

$$= \frac{J_1(\pi \rho)}{2\rho} \triangleq \text{jinc}(\rho) \quad (32)$$

$\Pi(r) \supset \text{jinc}(\rho)$

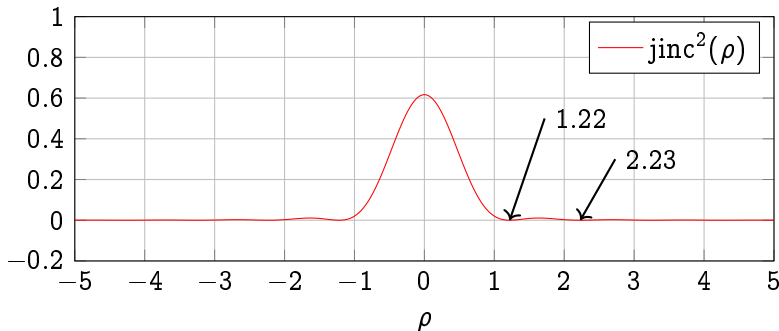
(33)

Fourier-Bessel transform of circ



use in imaging

The value 1.22 is very important when considering the resolution of an imaging system. We will also encounter $\text{jinc}^2(\rho)$, which is known as the Airy pattern. The inverse Fourier-Bessel transform is called the Chinese hat function, $\text{chat}(r)$.



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2D delta function

We can extend the concept of delta function to 2D, e.g.,

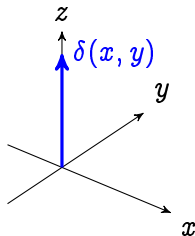
$${}^2\delta(x, y) \triangleq \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} {}^2\Pi\left(\frac{x}{\tau}, \frac{y}{\tau}\right) \quad (34)$$

As usual, we often ignore the “2” and overload the symbols.

This delta function has *unit volume*.

2D delta function

We can also have $\delta(x)$ and $\delta(y)$ in 2D.

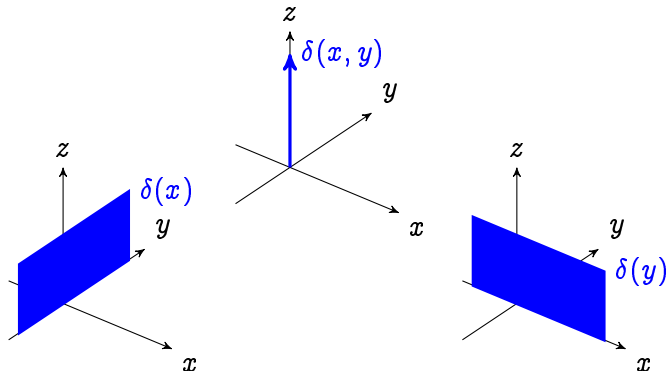


$$\delta(x, y) = \delta(x)\delta(y)$$

(35)

2D delta function

We can also have $\delta(x)$ and $\delta(y)$ in 2D.



$$\delta(x, y) = \delta(x)\delta(y)$$

(35)

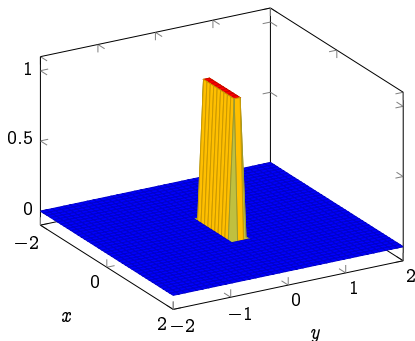
2D rect function

Another view of the 2D rect function:

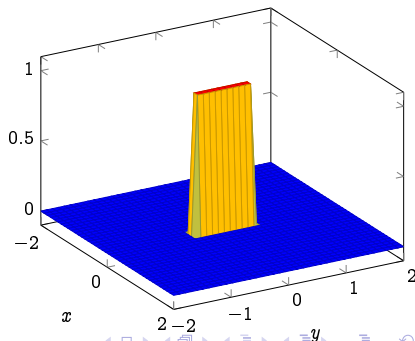
$$^2\Pi(x, y) = [\Pi(x)\delta(y)] ** [\Pi(y)\delta(x)] \quad (36)$$

(we often use $*$ instead of $**$ to denote 2D convolution)

$\Pi(x)\delta(y)$



$\Pi(y)\delta(x)$



delta in polar coordinates

We can also have delta functions in polar coordinates.

Does $\delta(x, y) = \delta(r)$?

delta in polar coordinates

We can also have delta functions in polar coordinates.

Does $\delta(x, y) = \delta(r)$?

$$\delta(x, y) = \frac{1}{\pi|r|} \delta(r) \quad (37)$$

(Reason: The proportional constant arises with the change of coordinates.)

Try to picture what $\delta(r - k)$ is, for a constant k .

line delta

Back to Cartesian coordinates, is

$$\delta(y - mx - c) \tag{38}$$

a “line delta” at the line $y = mx + c$?

line delta

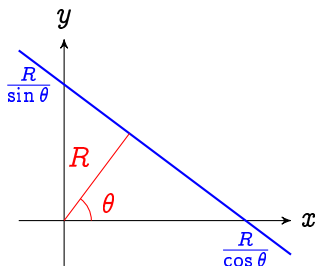
Back to Cartesian coordinates, is

$$\delta(y - mx - c) \quad (38)$$

a “line delta” at the line $y = mx + c$?

ANS: Need to be careful on the “strength” of the delta... the proper representation is

$$\delta(R - x \cos \theta - y \sin \theta) \quad (39)$$



shah in 2D

We can also have a shah function in 2D:

$$\mathbb{W}(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x - n, y - m) \quad (40)$$

Try to picture:

- $\mathbb{W}(x, y)$ (“bed of nails”)
- $\mathbb{W}(x)$ in 2D (“grating”)
- $\mathbb{W}(x)\delta(y)$ (“row of spikes”)

shah in 2D

How do we represent a “bed of nails” with a distance (X, Y) apart in the two directions? Remember $\delta(ax) = \frac{1}{|a|}\delta(x)$

shah in 2D

How do we represent a “bed of nails” with a distance (X, Y) apart in the two directions? Remember $\delta(ax) = \frac{1}{|a|}\delta(x)$

$$\delta(x - nX) = X\delta\left(\frac{x}{X} - n\right) \quad (41)$$

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x - nX, y - mY) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} XY\delta\left(\frac{x}{X} - n, \frac{y}{Y} - m\right) \quad (42)$$

$$= XY \sqcup \left(\frac{x}{X}, \frac{y}{Y}\right) \quad (43)$$

Therefore, the unit strength deltas would be

$$\frac{1}{XY} \sqcup \left(\frac{x}{X}, \frac{y}{Y}\right) \quad (44)$$

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delta

We can extend from 1D to 2D. What is the Fourier transform of $\delta(x, y)$?

delta

We can extend from 1D to 2D. What is the Fourier transform of $\delta(x, y)$?

$$\delta(x, y) \supset 1$$

(45)

delta

What is the Fourier transform of $\delta(x)$?

delta

What is the Fourier transform of $\delta(x)$?

$$\delta(x) \supset \delta(v) \quad (46)$$

(Hint: $\delta(x) = \delta(x) \cdot 1$)

cosine and sine

What is the Fourier transform of $\cos \pi(x + y)$?

cosine and sine

What is the Fourier transform of $\cos \pi(x + y)$?

$$\cos \pi(x + y) = \cos \pi x \cos \pi y - \sin \pi x \sin \pi y \quad (47)$$

$$\cos \pi x \cos \pi y = \left(\frac{1}{2} \left[\delta(u - \tfrac{1}{2}) + \delta(u + \tfrac{1}{2}) \right] \right) \left(\frac{1}{2} \left[\delta(v - \tfrac{1}{2}) + \delta(v + \tfrac{1}{2}) \right] \right) \quad (48)$$

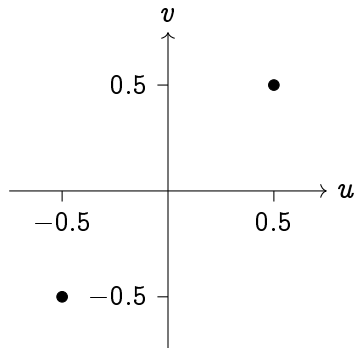
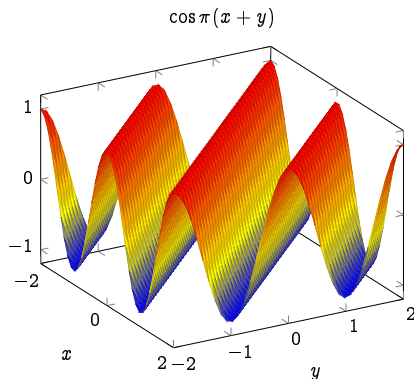
$$= \frac{1}{4} [\delta(u^-, v^-) + \delta(u^-, v^+) + \delta(u^+, v^-) + \delta(u^+, v^+)] \quad (49)$$

$$- \sin \pi x \sin \pi y = - \left(\frac{j}{2} \left[\delta(u + \tfrac{1}{2}) - \delta(u - \tfrac{1}{2}) \right] \right) \left(\frac{j}{2} \left[\delta(v + \tfrac{1}{2}) - \delta(v - \tfrac{1}{2}) \right] \right) \quad (50)$$

$$= \frac{1}{4} [\delta(u^-, v^-) - \delta(u^-, v^+) - \delta(u^+, v^-) + \delta(u^+, v^+)] \quad (51)$$

cosine and sine

$$\cos \pi(x + y) \supset \frac{1}{2} \left[\delta(u - \tfrac{1}{2}, v - \tfrac{1}{2}) + \delta(u + \tfrac{1}{2}, v + \tfrac{1}{2}) \right] \quad (52)$$



shah function

What is $\mathcal{F}\{\text{sh}(x, y)\}$?

shah function

What is $\mathcal{F}\{\mathbb{W}(x, y)\}$?

$$\mathbb{W}(x, y) \supset \mathbb{W}(u, v) \quad (53)$$

$$\mathbb{W}(x) \supset \mathbb{W}(u)\delta(v) \quad (54)$$

$$\mathbb{W}(x)\delta(y) \supset \mathbb{W}(u) \quad (55)$$