

Useful functions and their transforms (I)

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ELEC8501: The Fourier Transform and Its Applications
(First Semester, 2022-23)

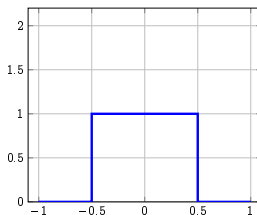
<https://www.eee.hku.hk/~elec8501>

- 1 Some useful functions
- 2 Fourier transform of some useful functions
- 3 The delta function
- 4 Fourier transform related to the delta function

(1) rect function

In Fourier transform, we often deal with piecewise linear functions. The simplest one is a *rect function*.

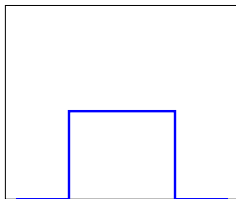
$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$



rect function

The rect function has a unit area. But we can *scale*, *shift*, and *stretch*, with

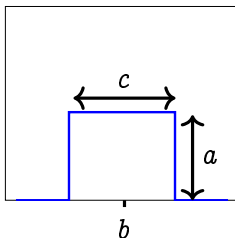
$$f(t) = a\Pi\left(\frac{t-b}{c}\right) \quad (2)$$



rect function

The rect function has a unit area. But we can *scale*, *shift*, and *stretch*, with

$$f(t) = a\Pi\left(\frac{t-b}{c}\right) \quad (2)$$



rect function

The rect function can be used to:

- 1 change integration limit:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) dt = \int_{-\infty}^{\infty} f(t) \Pi(t) dt \quad (3)$$

- 2 extract a segment of a function:

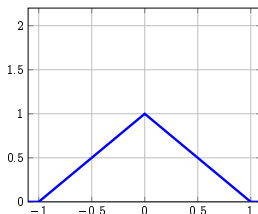
$$f(t) \Pi\left(\frac{t-b}{c}\right) \quad (4)$$

- 3 approximate a function as piecewise linear (Riemann integral)
- 4 interpolate a discrete signal (more later)

(2) triangle function

Next we have the *triangle function* (of unit area).

$$\Lambda(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$



Sketch $f(t) = a\Lambda\left(\frac{t-b}{c}\right)$.

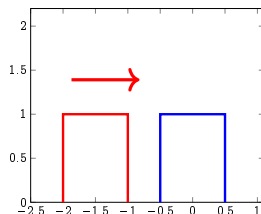
triangle function

A very important relationship:

$$\Pi(t) * \Pi(t) = \Lambda(t) \quad (6)$$

We use $*$ to represent convolution:

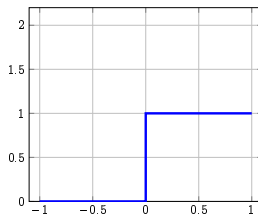
$$f(t) * g(t) \triangleq \int_{-\infty}^{\infty} f(u)g(x - u)du \quad (7)$$



(3) step function

Next, we have the Heaviside step function

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$



Sketch $f(t) = aH(t - b)$.

step function

- 1 Make a signal causal (kill the signal before $t = 0$)

$$g(t) = H(t)f(t) \quad (9)$$

- 2 Integrate a function:

$$H(t) * f(t) = \int_{-\infty}^{\infty} f(u) \underbrace{H(t-u)}_{=1 \text{ when } t-u > 0} du \quad (10)$$

$$= \int_{-\infty}^t f(u) du \quad (11)$$

Differentiating both sides gives:

$$\frac{d}{dt}[H(t) * f(t)] = f(t) \quad (12)$$

step function

- 1 How do you represent $\int_0^t f(u)du$?

step function

- ① How do you represent $\int_0^t f(u)du$?

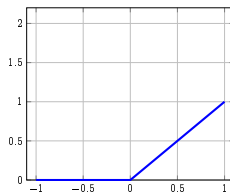
$$H(t) * [H(t)f(t)] = \int_0^t f(u)du \quad (13)$$

- ② We can create a rect from a step:

$$\Pi(t) = H(t + \frac{1}{2}) - H(t - \frac{1}{2}) \quad (14)$$

step function

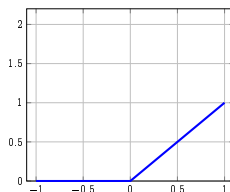
We may want to have a ramp function:



We are accustomed to calling it the ReLU: $f(t) = t^+ = \max(0, t)$

step function

We may want to have a ramp function:



We are accustomed to calling it the ReLU: $f(t) = t^+ = \max(0, t)$

- 1 Can use a step function to create a ramp:

$$f(t) = tH(t) \quad (15)$$

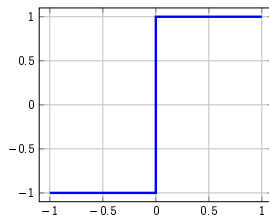
We also have $f(t) = \int_{-\infty}^t H(u)du$. Therefore,

$$f(t) = H(t) * H(t) \quad (16)$$

(4) signum function

We define the “sign” function (but pronounced as “signum”) as

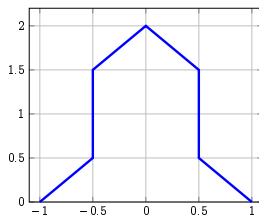
$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & \text{otherwise.} \end{cases} \quad (17)$$



$$\text{sgn}(t) = 2H(t) - 1 \quad (18)$$

Some exercises

- Express algebraically



- Sketch $H(t)e^{-t}$
- Sketch and show $\frac{d}{dt}\Lambda(t) = -\Pi\left(\frac{t}{2}\right)\text{sgn}(t)$
- Prove that $\Pi(x^2) = \Pi(x/\sqrt{2})$. Hint: How to interpret

$$\Pi(f(t)) = \begin{cases} 1 & |f(t)| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \quad (19)$$

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(1) rect function

Computer $\mathcal{F}\{\Pi(t)\}$ by direct integration (fill in the steps):

(1) rect function

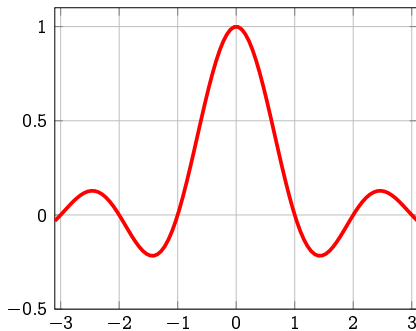
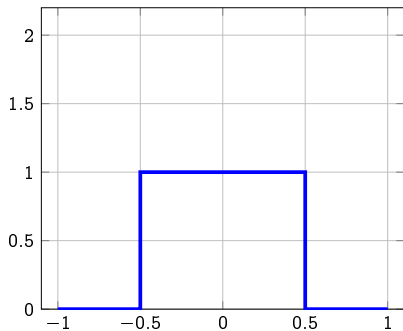
Computer $\mathcal{F}\{\Pi(t)\}$ by direct integration (fill in the steps):

$$\Pi(t) \supset \text{sinc}(s) \quad (20)$$

where $\text{sinc}(s) \triangleq \frac{\sin \pi s}{\pi s}$

rect and sinc

Try to label the graphs:



(2) triangle function

Computer $\mathcal{F}\{\Lambda(t)\}$ by direct integration (try it yourself):

(2) triangle function

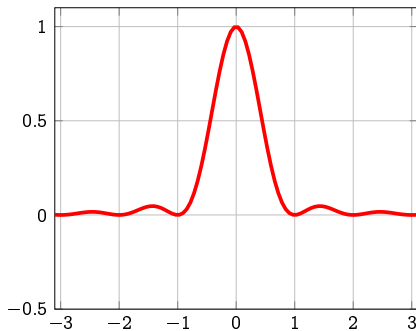
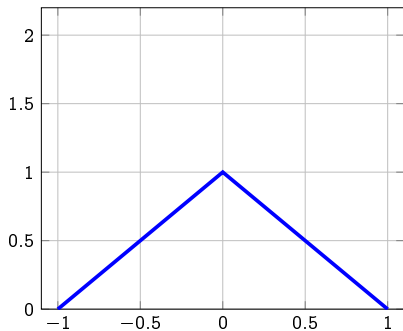
Computer $\mathcal{F}\{\Lambda(t)\}$ by direct integration (try it yourself):

$$\Lambda(t) \supset \text{sinc}^2(s) \quad (21)$$

(there is a better way to do this than direct integration)

triangle and sinc squared

Try to label the graphs:



(3) step function

$$H(t) \supset \frac{1}{2}\delta(s) - \frac{j}{2\pi s} \quad (22)$$

We will defer discussion of $\delta(s)$ in the meantime.

It's closely related to the signum function, so let's look into that first.

(4) signum function

Computer $\mathcal{F}\{\text{sgn}(t)\}$ by direct integration (try it yourself):

(4) signum function

Computer $\mathcal{F}\{\text{sgn}(t)\}$ by direct integration (try it yourself):

How do we deal with:

$$\mathcal{F}\{\text{sgn}(t)\} = j \left[\frac{\cos(2\pi st)}{2\pi s} \right]_{t=-\infty}^{t=\infty} - \frac{j}{\pi s} \quad (23)$$

It fails a sufficient condition for the existence of Fourier transform:

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

Fourier transform in the limit

A 3-step approach called “Fourier transform in the limit”:

- ① Define $f_N(t)$ such that $\lim_{N \rightarrow \infty} f_N(t) = f(t)$
- ② Compute $F_N(s) = \mathcal{F}\{f_N(t)\}$
- ③ Define $F(s) \triangleq \lim_{N \rightarrow \infty} F_N(s)$

An example:

$$f_N(t) = \begin{cases} e^{-t/N} & t \geq 0 \\ -e^{t/N} & \text{otherwise.} \end{cases} \quad (24)$$

(Sketch it yourself.)

Fourier transform in the limit

- ① Check $\lim_{N \rightarrow \infty} f_N(t) = \text{sgn}(t)$
- ② Compute $F_N(s) = \mathcal{F}\{f_N(t)\}$ to obtain

$$F_N(s) = \frac{1}{\frac{1}{N} + j2\pi s} - \frac{1}{\frac{1}{N} - j2\pi s} \quad (25)$$

- ③ Therefore,

$$F(s) = \lim_{N \rightarrow \infty} \left(\frac{1}{\frac{1}{N} + j2\pi s} - \frac{1}{\frac{1}{N} - j2\pi s} \right) \quad (26)$$

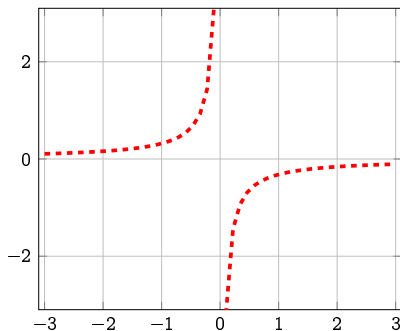
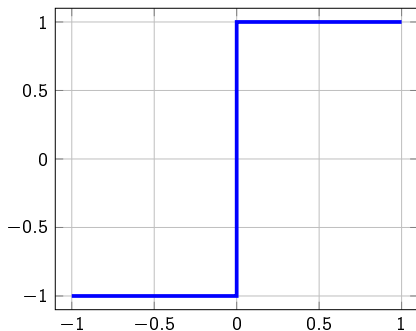
$$= \frac{1}{j2\pi s} + \frac{1}{j2\pi s} \quad (27)$$

$$= -\frac{j}{\pi s} \quad (28)$$

signum function

$$\text{sgn}(t) \supset -\frac{j}{\pi s}$$

(29)



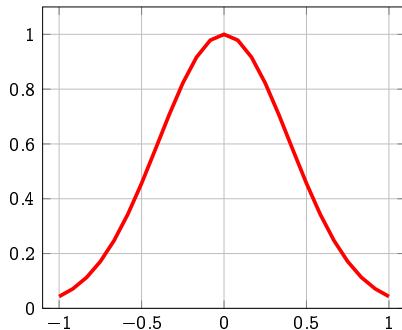
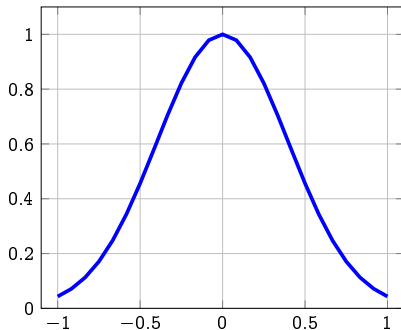
(5) Gaussian function

Computer $\mathcal{F}\{e^{-\pi x^2}\}$ by direct integration (try it yourself):

(5) Gaussian function

Computer $\mathcal{F}\{e^{-\pi x^2}\}$ by direct integration (try it yourself):

$$e^{-\pi x^2} \supset e^{-\pi s^2} \quad (30)$$



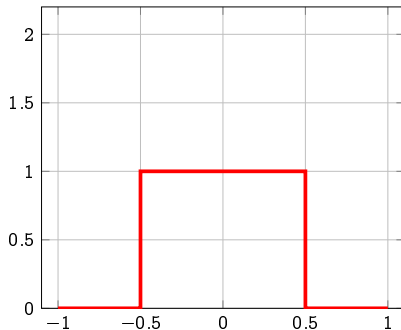
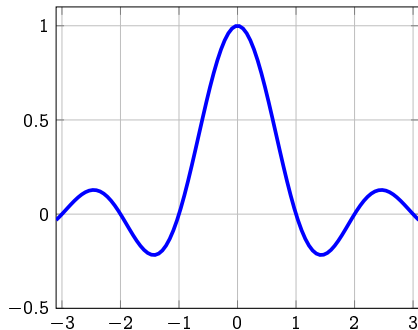
(6) sinc function

Computer $\mathcal{F}\{\text{sinc}(x)\}$ by direct integration (try it yourself):

(6) sinc function

Computer $\mathcal{F}\{\text{sinc}(x)\}$ by direct integration (try it yourself):

$$\text{sinc}(x) \supset \Pi(s) \quad (31)$$



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an impulse

To understand the behavior of a system, we often want to hit it with a short pulse (input) and observe what happens (output)

This short pulse is an “impulse”

What we hope to have: a function where

$$\begin{cases} f(t) = 0 & t \neq 0 \\ \int_{-\infty}^{\infty} f(t) dt = 1 \end{cases} \quad (32)$$

but it is not possible to have such a function!

delta function

So, we define the (Dirac) delta function as a *generalized function*.

$$\delta(t) \triangleq \lim_{\tau \rightarrow 0} \frac{1}{\tau} \Pi \left(\frac{t}{\tau} \right) \quad (33)$$

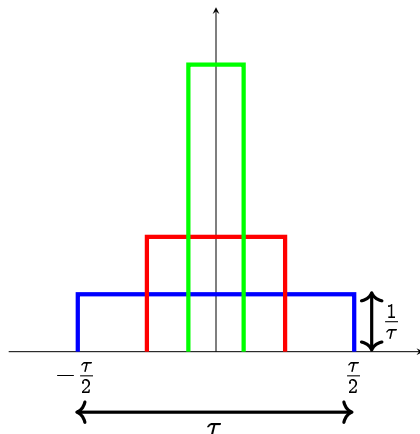
$$\delta(t) \triangleq \lim_{\tau \rightarrow 0} \frac{1}{\tau} \Lambda \left(\frac{t}{\tau} \right) \quad (34)$$

$$\delta(t) \triangleq \lim_{\tau \rightarrow 0} \frac{1}{\tau} e^{-\pi t^2 / \tau^2} \quad (35)$$

$$\delta(t) \triangleq \lim_{\tau \rightarrow 0} \frac{1}{\tau} \text{sinc} \left(\frac{t}{\tau} \right) \quad (36)$$

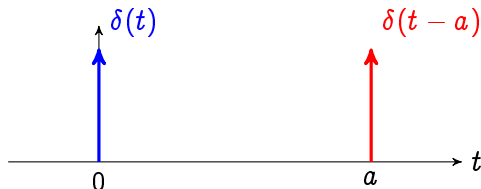
delta function

The delta function has unit area, and is zero everywhere except at the origin. For example, $\delta(t) \triangleq \lim_{\tau \rightarrow 0} \frac{1}{\tau} \Pi\left(\frac{t}{\tau}\right)$:



delta function properties

We draw $\delta(t)$ with an arrow:



The defining characteristic is the *sifting property*:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(t) f(t) dt &= \int_{-\infty}^{\infty} \left[\lim_{\tau \rightarrow 0} \frac{1}{\tau} \Pi \left(\frac{t}{\tau} \right) \right] f(t) dt \\ &= f(0) \end{aligned} \quad (37)$$

delta function properties

From the sifting properties, we can derive:

$$\int_{-\infty}^{\infty} \delta(t - a) f(t) dt = f(a) \quad (38)$$

$$\delta(t) * f(t) = \int_{-\infty}^{\infty} f(u) \delta(t - u) du = f(t) \quad (39)$$

$$\mathcal{F} \{ \delta(t) \} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi st} dt = 1 \quad (40)$$

delta function properties

It is important to keep in mind that while $\delta(t)$ does not have a “height” (at $t = 0$), it has an “area”, and therefore can be scaled.

Example: Express $\delta(at)$ for $a > 0$.

delta function properties

It is important to keep in mind that while $\delta(t)$ does not have a “height” (at $t = 0$), it has an “area”, and therefore can be scaled.

Example: Express $\delta(at)$ for $a > 0$.

$$\begin{aligned}\delta(at) &\triangleq \lim_{\tau \rightarrow 0} \frac{1}{\tau} \Pi\left(\frac{at}{\tau}\right) \\ &= \frac{1}{a} \lim_{\tau \rightarrow 0} \frac{a}{\tau} \Pi\left(\frac{at}{\tau}\right) \\ &= \frac{1}{a} \lim_{\tau' \rightarrow 0} \frac{1}{\tau'} \Pi\left(\frac{t}{\tau'}\right)\end{aligned}$$

where $\tau' = \tau/a$. Generalizing,

$$\delta(at) = \frac{1}{|a|} \delta(t) \quad (41)$$

impulse and step

We can relate the delta function and the step function as follows. Since

$$\int_{-\infty}^t \delta(u) du = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \quad (42)$$

ignoring what happens at $t = 0$, we can say

$$\int_{-\infty}^t \delta(u) du = H(t) \quad (43)$$

Furthermore,

$$\delta(t) = \frac{d}{dt} H(t) \quad (44)$$

Note that by calculus, $\frac{d}{dt} 2H(t) = 2\delta(t)$.

delta function properties

Some further relationships:

- $\delta(-t) = \delta(t)$
- $f(t)\delta(t) = f(0)\delta(t)$
- $t\delta(t) = 0$
- $\mathcal{F}^{-1}\{1\} = \delta(t) = \int_{-\infty}^{\infty} e^{j2\pi st} ds$

derivative of delta

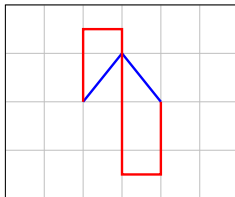
There are occasions (such as in the chain rule) that we need to calculate

$$\delta'(t) = \frac{d}{dt}\delta(t) \quad (45)$$

We can define it with the derivative-sifting property

$$\delta'(t) * f(t) = \int_{-\infty}^{\infty} \delta'(t-u)f(u)du = f'(t) \quad (46)$$

$$\frac{1}{\tau}\Lambda\left(\frac{t}{\tau}\right) \rightarrow \frac{1}{\tau^2}\Pi\left(\frac{t}{2\tau}\right)\text{sgn}(t)$$



delta derivative properties

Some further relationships:

- $\delta'(-t) = -\delta'(t)$
- $t\delta'(t) = -\delta(t)$
- $f(t)\delta'(t) = f(0)\delta'(t) - f'(0)\delta(t)$. Proof:

$$\frac{d}{dt}f(t)\delta(t) = f'(t)\delta(t) + f(t)\delta'(t)$$

$$\frac{d}{dt}f(0)\delta(t) = f'(0)\delta(t) + f(t)\delta'(t)$$

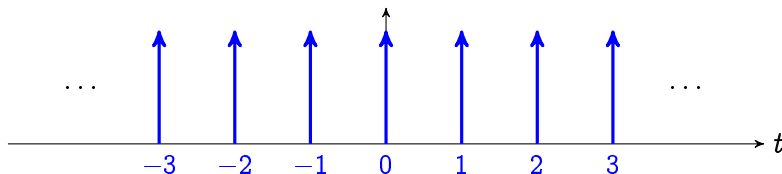
$$f(0)\delta'(t) - f'(0)\delta(t) = f(t)\delta'(t)$$

- $\int_{-\infty}^{\infty} t\delta'(t)dt = -1$
- $\delta^{(n)}(t) * f(t) = f^{(n)}(t)$

shah function

Define the shah (comb) function as an infinite train of delta functions:

$$\mathfrak{W}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n) \quad (47)$$



shah function

The shah function is most useful in *sampling* and *replication*:

$$\mathfrak{W}(t)f(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t-n) \quad (48)$$

$$\mathfrak{W}(t) * f(t) = \sum_{n=-\infty}^{\infty} f(t-n) \quad (49)$$

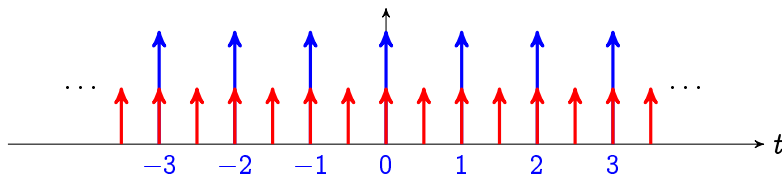
shah function

What is $\mathbb{W}(2t)$?

shah function

What is $\mathfrak{W}(2t)$?

$$\mathfrak{W}(2t) = \sum_{n=-\infty}^{\infty} \delta(2t - n) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{2}\right) \quad (50)$$



To make each impulse of unit strength, we need $2 \mathfrak{W}(2t)$.

shah function

Some more properties (assume n is an integer):

- $\mathbb{W}(at) = \frac{1}{|a|} \sum_{n=-\infty}^{\infty} \delta(t - \frac{n}{a})$

- $\mathbb{W}(-t) = \mathbb{W}(t)$

- $\mathbb{W}(t + n) = \mathbb{W}(t)$

- $\mathbb{W}(t - \frac{1}{2}) = \mathbb{W}(t + \frac{1}{2})$

- $\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \mathbb{W}(t) dt = 1$

- $\mathbb{W}(t) = 0$ for $t \neq n$

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cosine and sine

What is the Fourier transform of $\cos \pi t$?

$$\mathcal{F}\{\cos \pi t\} = \mathcal{F}\left\{\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right\} \quad (51)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{j\pi t} e^{-j2\pi st} dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j\pi t} e^{-j2\pi st} dt \quad (52)$$

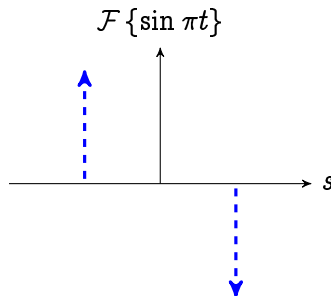
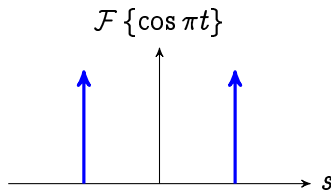
$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi(s-\frac{1}{2})t} dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi(s+\frac{1}{2})t} dt \quad (53)$$

$$= \frac{1}{2} \delta(s - \frac{1}{2}) + \frac{1}{2} \delta(s + \frac{1}{2}) \quad (54)$$

cosine and sine

$$\cos \pi t \supset \frac{1}{2} \left[\delta\left(s - \frac{1}{2}\right) + \delta\left(s + \frac{1}{2}\right) \right] \quad (55)$$

$$\sin \pi t \supset \frac{j}{2} \left[\delta\left(s + \frac{1}{2}\right) - \delta\left(s - \frac{1}{2}\right) \right] \quad (56)$$



cosine and sine

The same “pair of impulses” exist for different sinusoidal frequencies, e.g.,

$$\cos t \supset \frac{1}{2} \left[\delta\left(s - \frac{1}{2\pi}\right) + \delta\left(s + \frac{1}{2\pi}\right) \right] \quad (57)$$

$$\sin t \supset \frac{j}{2} \left[\delta\left(s + \frac{1}{2\pi}\right) - \delta\left(s - \frac{1}{2\pi}\right) \right] \quad (58)$$

shah function

What is $\mathcal{F}\{\text{sh}(t)\}$?

shah function

What is $\mathcal{F}\{\mathfrak{W}(t)\}$?

Since $\mathcal{F}\{\delta(t - n)\} = \int_{-\infty}^{\infty} \delta(t - n)e^{-j2\pi st} dt = e^{-j2\pi ns}$, we have

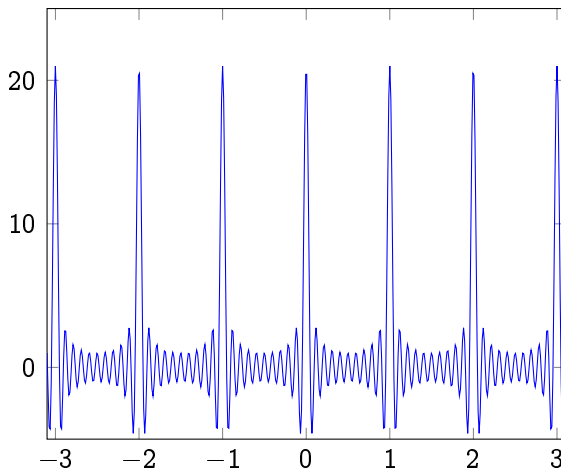
$$\mathcal{F}\{\mathfrak{W}(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - n)\right\} = \sum_{n=-\infty}^{\infty} e^{-j2\pi ns} \quad (59)$$

This function “blows up” when s takes on integer values, but is zero otherwise.

Is it $\mathfrak{W}(s)$? Or a scaled version of it?

shah function

This is a plot of $\sum_{n=-10}^{10} e^{-j2\pi ns}$



shah function

We notice that $F(s) = \sum_{n=-\infty}^{\infty} e^{-j2\pi ns}$ is a *periodic function with period 1* since $F(s) = F(s+1) = F(s+2)$ etc.

We make use of Fourier series of a function $G(t)$ with period L :

$$G(s) = \sum_{n=-\infty}^{\infty} \alpha_n e^{j2\pi \frac{n}{L}s} \quad (60)$$

$$\alpha_n = \frac{1}{L} \int_{-L/2}^{L/2} G(s) e^{-j2\pi \frac{n}{L}s} ds \quad (61)$$

Now we consider $L = 1$.

shah function

- Since $F(s) = \sum_{n=-\infty}^{\infty} e^{-j2\pi ns} = \sum_{n=-\infty}^{\infty} e^{j2\pi ns}$, comparing with Eq. (60), we have

$$\alpha_n = 1 \quad (62)$$

- Let $\hat{F}(s) = F(s)\Pi(s)$, i.e., only one period. Eq. (61) now says:

$$1 = \int_{-1/2}^{1/2} F(s)e^{-j2\pi ns} ds \quad (63)$$

$$= \int_{-\infty}^{\infty} F(s)\Pi(s)e^{-j2\pi ns} ds \quad (64)$$

So $\hat{F}(s)$ is the (inverse) Fourier transform of 1, which means $\hat{F}(s) = \delta(s)$.

shah function

- We can write $F(s)$ as a summation of (shifted) $\hat{F}(s)$:

$$F(s) = \sum_{n=-\infty}^{\infty} \hat{F}(s - n) = \sum_{n=-\infty}^{\infty} \delta(s - n) = \mathfrak{W}(s) \quad (65)$$

$$\mathfrak{W}(t) \supset \mathfrak{W}(s) \quad (66)$$