

Fundamentals of Fourier transform

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ELEC8501: The Fourier Transform and Its Applications
(First Semester, 2022-23)

<https://www.eee.hku.hk/~elec8501>

- 1 The Fourier transform
- 2 Symmetry arguments and conjugate function
- 3 Visualizing Fourier transforms
- 4 Interpreting Fourier transforms

Equations

The (continuous) Fourier transform equations:

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st} dt \quad (1)$$

$$f(t) = \int_{-\infty}^{\infty} F(s)e^{j2\pi st} ds \quad (2)$$

- Can write $F(s) = \mathcal{F}\{f(t)\}$ and $f(t) = \mathcal{F}^{-1}\{F(s)\}$
- Normally say $f(t)$ is in the *time domain*, $F(s)$ is in the *frequency domain* or *Fourier domain*
- Can also write $f(t) \supset F(s)$
- i or j ? $F(s)$ or $\hat{f}(s)$?
- Use $\omega = 2\pi s$? Symmetry in the two equations?

Equations

More critical (mathematical) questions:

- Integral exists?
- How do I know

$$f(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \right] e^{j2\pi st} ds \quad (3)$$

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“The question of the existence of transforms may safely be ignored when the function to be transformed is an accurately specified description of a physical quantity. Physical possibility is a valid sufficient condition for the existence of a transform.” (Quote from Prof. Bracewell) — note that e.g., $f(t) = \sin t$ is not physical...

Related equations

Eq. (1) and (2) are known as continuous Fourier transform (CFT).

Discrete-time Fourier transform (DTFT):

$$F(s) = \sum_{n=-\infty}^{\infty} f[n]e^{-j2\pi sn} \quad (4)$$

Discrete Fourier transform (DFT):

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n]e^{-j2\pi kn/N} \quad (5)$$

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Any function $f(t)$ can be split unambiguously into odd and even parts:

$$f(t) = e(t) + o(t) \quad (8)$$

(how do you show that this decomposition is unique?)

Symmetry

Calculating $e(t)$ and $o(t)$ from $f(t)$:

$$e(t) = \frac{1}{2} [f(t) + f(-t)] \quad (9)$$

$$o(t) = \frac{1}{2} [f(t) - f(-t)] \quad (10)$$

(try some examples yourselves)

Symmetry

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \quad (11)$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} e(t) \cos(2\pi st) dt - j \int_{-\infty}^{\infty} e(t) \sin(2\pi st) dt \\ &\quad + \int_{-\infty}^{\infty} o(t) \cos(2\pi st) dt - j \int_{-\infty}^{\infty} o(t) \sin(2\pi st) dt \end{aligned} \quad (12)$$

$$= \int_{-\infty}^{\infty} e(t) \cos(2\pi st) dt - j \int_{-\infty}^{\infty} o(t) \sin(2\pi st) dt \quad (13)$$

If $f(t)$ is even? Or $f(t)$ is real and even? Or $f(t)$ is real?

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If $f(t)$ is even? Or $f(t)$ is real and even? Or $f(t)$ is real?

even $\supset$ even

(14)

odd $\supset$ odd

(15)

Symmetry

Breaking down further with real and imaginary parts:

$$f(t) = \operatorname{Re}\{e(t)\} + j\operatorname{Im}\{e(t)\} + \operatorname{Re}\{o(t)\} + j\operatorname{Im}\{o(t)\}$$

$$F(s) = \operatorname{Re}\{E(s)\} + j\operatorname{Im}\{E(s)\} + \operatorname{Re}\{O(s)\} + j\operatorname{Im}\{O(s)\}$$

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real and even $\supset$ real and even

(16)

imaginary and even $\supset$ imaginary and even

(17)

real and odd $\supset$ imaginary and odd

(18)

imaginary and odd $\supset$ real and odd

(19)

Conjugate

The (complex) conjugate of a function $f(t)$ is $f^*(t)$. If $f(t) \rightrightarrows F(s)$, what does its conjugate transform to?

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$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \quad (20)$$

$$F^*(s) = \int_{-\infty}^{\infty} f^*(t) e^{j2\pi st} dt \quad (21)$$

$$F^*(-s) = \int_{-\infty}^{\infty} f^*(t) e^{-j2\pi(-s)t} dt \quad (22)$$

So, in general,

$$f^*(t) \rightrightarrows F^*(-s) \quad (23)$$

Conjugate

Special case: $f(t)$ is real.

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If $f(t)$ is real, then $f(t) = f^*(t)$, and therefore $F(s) = F^*(-s)$. The real part therefore is even, and the imaginary part is odd. We say that such a function is *Hermitian*.

$$\text{real} \supset \text{Hermitian} \quad (24)$$

$$\text{odd} \supset \text{anti-Hermitian} \quad (25)$$

Conjugate

What does $f(t) + f^*(t)$ transform to?

Conjugate

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$f(t) + f^*(t) \supset F(s) + F^*(-s)$. Therefore,

$$\operatorname{Re}\{f(t)\} \supset \frac{1}{2}(F(s) + F^*(-s)) \quad (26)$$

How about $\operatorname{Im}\{f(t)\}$?

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Mental picture

You can use a computer to calculate (numerical approximation of) various Fourier transforms. So why do we bother with:

- doing it by “paper and pencil”?
- doing it mentally?

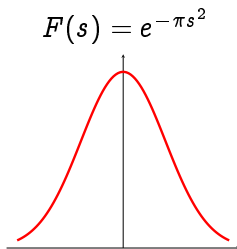
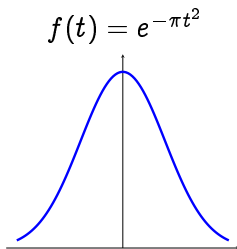
Example 1

What is the Fourier transform of a Gaussian function (ignoring the normalization) $f(t) = e^{-\pi t^2}$?

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ANS: It is a real and even function, so its Fourier transform is also real and even. We'll have to “do the math” to get to the actual answer, which is $F(s) = e^{-\pi s^2}$. But because both domains are real (only), we can easily plot and think about them:



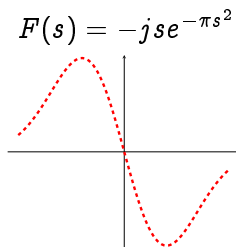
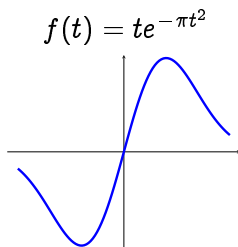
Example 2

What is the Fourier transform of $f(t) = te^{-\pi t^2}$?

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What is the Fourier transform of $f(t) = te^{-\pi t^2}$?

ANS: It is a real and odd function, so its Fourier transform is imaginary and odd. We can plot only the imaginary part in the Fourier transform:



Example 3

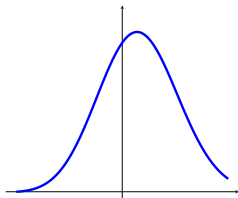
What is the Fourier transform of $f(t) = (1 + t)e^{-\pi t^2}$?

Example 3

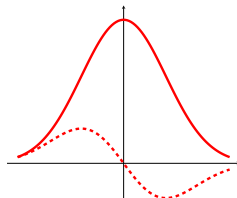
What is the Fourier transform of $f(t) = (1 + t)e^{-\pi t^2}$?

ANS: It is a real function (but neither odd nor even), so its Fourier transform is Hermitian (complex). We can *choose* to plot both the real and imaginary parts on the same diagram: (Sometimes, we may choose to plot only the magnitude)

$$f(t) = (1 + t)e^{-\pi t^2}$$



$$F(s) = (1 - js)e^{-\pi s^2}$$



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Meaning of Fourier transform

Remember from Eq. (2) that

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{j2\pi st} ds \quad (27)$$

So, we *decompose* $f(t)$ into a sum (integration) of exponentials (sines and cosines at various frequencies), where each $F(s)$ serves as the *weight*

Therefore, $F(s)$ gives us a frequency (spectrum) view of the signal