

Fourier transform applications: Tomographic imaging

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ELEC8501: The Fourier Transform and Its Applications

(First Semester, 2022–23)

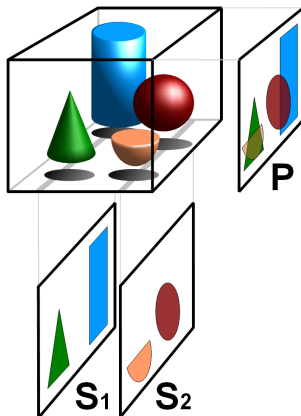
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Tomography problem statement

Situation

A common problem, such as in medical imaging (computed tomography):



Source: Wikipedia entry on “Tomography”

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Tomography problem statement

1 Tomography problem statement

2 Projection and slice

3 Circularly symmetric function

4 Filtered backprojection

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Tomography problem statement

Situation

For simplicity, let us fix at a certain vertical position (z).

Questions:

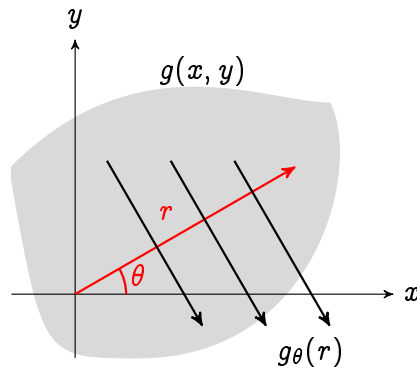
- With “enough” (1D) projection at different angles, can we reconstruct the full (2D) signal?
- What is enough?
- What is the reconstruction algorithm?

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Reconstruction from projections

To put it in mathematical terms:

Given $g_\theta(r)$ for a range of θ , reconstruct $g(x, y)$.



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Reconstruction from projections

We have seen this setup in studying “line delta” in Lecture 3. Each line of projection is

$$r = x \cos \theta + y \sin \theta \quad (1)$$

So how about this:

1 Define

$$b_\theta(x, y) = g_\theta(x \cos \theta + y \sin \theta) \quad (2)$$

(this is a *backprojection* step)

2 Compute

$$g(x, y) = \int_0^\pi b_\theta(x, y) d\theta \quad (3)$$

(this is a *summation* step)

Turns out we can do better... with Fourier transform!

Projection

Consider a vertical projection, i.e., $\theta = 0$.

$$g_0(x) = \int_{-\infty}^{\infty} g(x, y) dy \quad (4)$$

This has a (1D) Fourier transform:

$$\begin{aligned} g_0(x) &\supset \int_{-\infty}^{\infty} g_0(x) e^{-j2\pi ux} dx \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(x, y) dy \right] e^{-j2\pi ux} dx \\ &= G(u, 0) \end{aligned} \quad (5)$$

Projection

The very simple derivations above have a profound implication:

Projection of 2D $\xrightarrow{\mathcal{F}_{1D}}$ Slice of 2D in the Fourier domain

This is known as the projection-slice theorem, or Fourier-slice theorem.

But there shouldn't be anything special about $\theta = 0 \dots$ how about other angles?

Radon transform

$$g_{\theta}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \delta(r - x \cos \theta - y \sin \theta) dx dy \quad (9)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{g}(\hat{x}, \hat{y}) \delta(r - \hat{x}) d\hat{x} d\hat{y} \quad (10)$$

$$= \int_{-\infty}^{\infty} \hat{g}(r, \hat{y}) d\hat{y} \quad (11)$$

Then,

$$g_{\theta}(\hat{x}) \supset \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \hat{g}(\hat{x}, \hat{y}) d\hat{y} \right] e^{-j2\pi s \hat{x}} d\hat{x} \quad (12)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{g}(\hat{x}, \hat{y}) e^{-j2\pi s \hat{x}} d\hat{x} d\hat{y} \quad (13)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi s(x \cos \theta + y \sin \theta)} dx dy \quad (14)$$

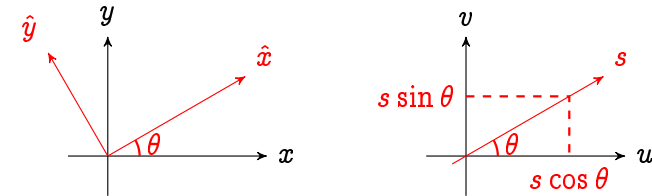
$$= G(s \cos \theta, s \sin \theta) \quad (15)$$

Radon transform

The Radon transform of a 2D function (at angle θ) is given by

$$g_{\theta}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \delta(r - x \cos \theta - y \sin \theta) dx dy \quad (6)$$

We want to show that $\mathcal{F}\{g_{\theta}(r)\} = \text{slice of 2D Fourier at angle } \theta$



$$\hat{x} = x \cos \theta + y \sin \theta \quad (7)$$

$$\hat{y} = -x \sin \theta + y \cos \theta \quad (8)$$

Projection-slice theorem

We now have a more general statement:

Projection (at θ) of 2D $\xrightarrow{\mathcal{F}_{1D}}$ Slice of 2D (at θ) in Fourier

Let us now see how we can make use of this.

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2 Projection and slice

3 Circularly symmetric function

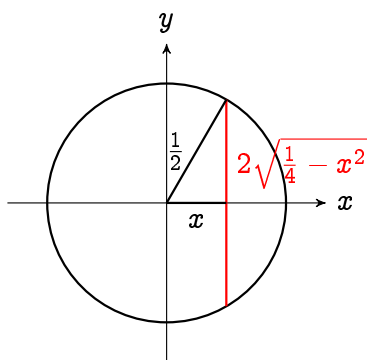
4 Filtered backprojection

Abel transform

The projection of a circularly symmetric function $g(r)$ is given by the Abel transform:

$$g_A(x) = \mathcal{A}\{g(r)\} = 2 \int_x^\infty \frac{g(r)}{\sqrt{r^2 - x^2}} r dr \quad (16)$$

Example:



$$\mathcal{A}\{\Pi(r)\} = \sqrt{1 - 4x^2} \Pi(x)$$

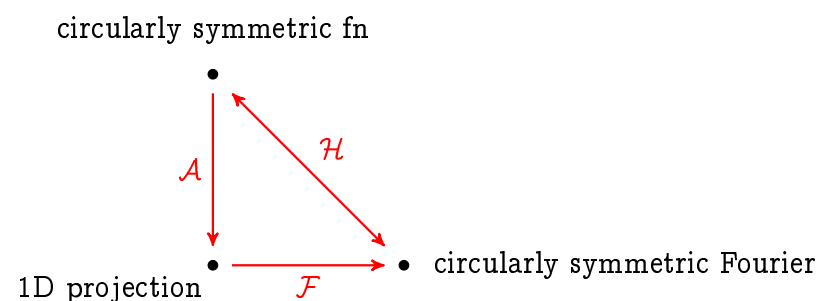
Circularly symmetric function

Consider a special case: what if $g(x, y)$ is circularly symmetric?

From Lecture 3: 2D Fourier transform of a circularly symmetric function is also circularly symmetric. We want to combine this fact with the Fourier-slice theorem to show that *one projection is enough*.

Abel-Hankel-Fourier ring

Projection-slice theorem says: A circularly symmetric function is related to its “projection (Abel) then Fourier” by a Hankel transform

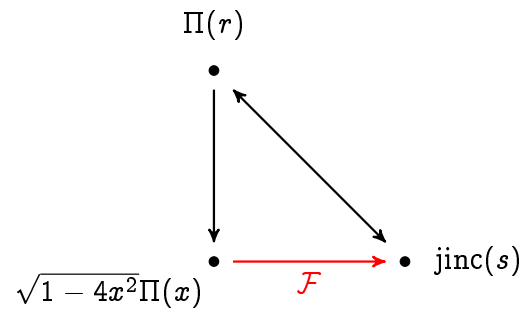


$$\mathcal{H} = \mathcal{F}\mathcal{A} \quad (17)$$

$$\mathcal{H}\mathcal{F}\mathcal{A} = \text{identity} \quad (18)$$

Abel-Hankel-Fourier ring

Examples:



1 Tomography problem statement

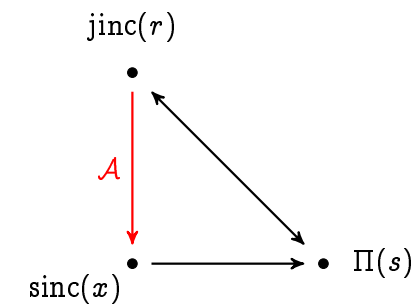
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Abel-Hankel-Fourier ring

Examples:



Non-symmetric function

For functions that are not necessarily circularly symmetric, one projection is not enough. We will need to take projections at different angles. But how do we combine them?

Remember that we have data $G_\theta(s) \triangleq G(s \cos \theta, s \sin \theta)$.

Tomographic reconstruction

From the inverse Fourier transform formula:

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u, v) e^{j2\pi(xu + yv)} du dv \quad (19)$$

$$= \int_0^{2\pi} \int_0^{\infty} G_{\theta}(s) e^{j2\pi(s \cos \theta \cdot x + s \sin \theta \cdot y)} s ds d\theta \quad (20)$$

$$= \int_0^{\pi} \int_0^{\infty} G_{\theta}(s) e^{j2\pi s(x \cos \theta + y \sin \theta)} s ds d\theta \\ + \int_0^{\pi} \int_0^{\infty} G_{\theta+\pi}(s) e^{j2\pi s(x \cos(\theta+\pi) + y \sin(\theta+\pi))} s ds d\theta \quad (21)$$

We note that $G_{\theta+\pi}(s) = G_{\theta}(-s)$, and let $\hat{s} = -s$.

Tomographic reconstruction

This means we have four steps:

- ④ From the captured projection signal $g_{\theta}(\hat{x})$ (i.e., Radon transform), compute

$$G_{\theta}(s) = \mathcal{F} \{g_{\theta}(\hat{x})\} \quad (25)$$

- ② Multiply in the frequency domain (i.e., filter):

$$\tilde{G}_{\theta}(s) = |s| G_{\theta}(s) \quad (26)$$

- ③ Compute the inverse Fourier transform, and backprojection:

$$\tilde{g}_{\theta}(\hat{x}) = \mathcal{F}^{-1} \{ \tilde{G}_{\theta}(s) \} \quad (27)$$

$$\tilde{g}(x, y) = \tilde{g}_{\theta}(x \cos \theta + y \sin \theta) \quad (28)$$

Tomographic reconstruction

Continuing:

$$g(x, y) = \int_0^{\pi} \int_0^{\infty} G_{\theta}(s) e^{j2\pi s(x \cos \theta + y \sin \theta)} s ds d\theta \\ + \int_0^{\pi} \int_0^{\infty} G_{\theta}(\hat{s}) e^{j2\pi \hat{s}(x \cos \theta + y \sin \theta)} (-\hat{s}) (-d\hat{s}) d\theta \quad (22)$$

$$= \int_0^{\pi} \left[\int_{-\infty}^{\infty} |s| G_{\theta}(s) e^{j2\pi s(x \cos \theta + y \sin \theta)} ds \right] d\theta \quad (23)$$

$$= \int_0^{\pi} \left[\int_{-\infty}^{\infty} |s| G_{\theta}(s) e^{j2\pi s \hat{x}} ds \right] d\theta \quad (24)$$

Tomographic reconstruction

- ④ Sum for all angle:

$$g(x, y) = \int_0^{\pi} \tilde{g}(x, y) d\theta \quad (29)$$

This is known as the *filtered backprojection (FBP)* method.

The “filtered” step (i.e., step 2) is what sets it apart from the naïve algorithm at the beginning of this lecture.