Sampling: Relating continuous with discrete

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ELEC8501: The Fourier Transform and Its Applications
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https://www.eee.hku.hk/~elec8501

1 Sampling in time ↔ replication in frequency

2 Replication in time ↔ sampling in frequency

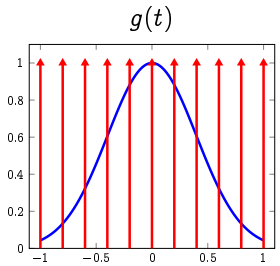
3 Sampling in time and sampling in frequency

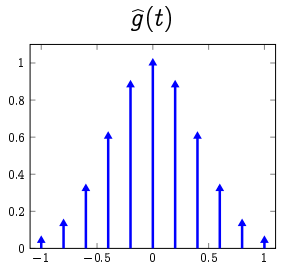
Sampling

Assume we start with a function $g(t)$. Let its sampled version be $\widehat{g}(t)$.

$$\boxed{\widehat{g}(t) = \sum_{n=-\infty}^{\infty} g(n\tau)\delta(t - n\tau) = g(t)\left[\frac{1}{\tau}\,\text{Ш}\left(\frac{t}{\tau}\right)\right]} \tag{1}$$

We call τ the sampling interval.





bandlimited

We call a signal *bandlimited* if:

$$\boxed{G(s) = 0 \quad \text{for} \quad |s| > s_c} \tag{2}$$

where s_c is the *cutoff frequency*.

Very important: A bandlimited signal cannot be time-limited.

sampling theorem

Shannon-Whittaker-Kotelnikov-Nyquist sampling theorem:

1. Bandlimited signals are completely determined by a sufficient number of samples
2. We have a reconstruction/interpolation algorithm

sampling theorem

We consider the Fourier transform of the sampled signal:

$$\hat{g}(t) = g(t)\left[\frac{1}{\tau}\,\text{ш}\left(\frac{t}{\tau}\right)\right] \tag{3}$$

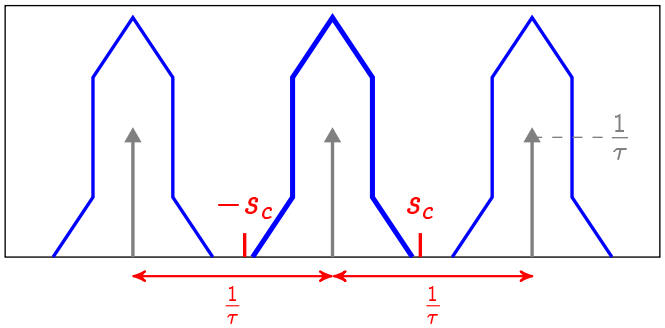
$$\hat{G}(s) = G(s) * \text{ш}(\tau s) \tag{4}$$

Note:

- $\text{ш}(\tau s)$ does not have unit height, but $1/\tau$
- When τ is small, then $1/\tau$ is large, which means the replicas are far away.

sampling theorem

A pictoral view (frequency domain):

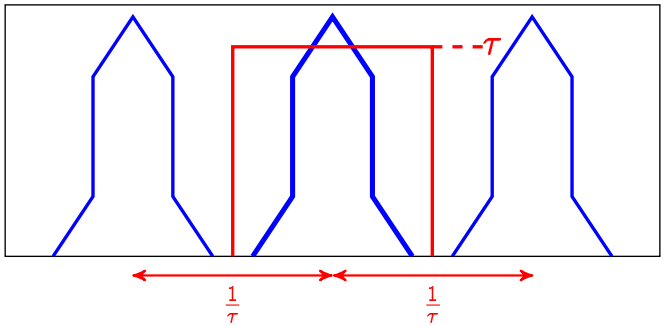


For a bandlimited function, we want $s_c < \frac{1}{2} \cdot \frac{1}{\tau}$ for no overlap.

$$\boxed{\underbrace{\frac{1}{\tau}}_{\text{sampling rate}} > \underbrace{2s_c}_{\text{Nyquist rate (bandwidth)}}} \tag{5}$$

sampling theorem

When there is no overlap, we can easily reconstruct:



$$\boxed{\begin{aligned} \tilde{G}(s) &\triangleq \hat{G}(s) \cdot \tau\Pi(\tau s) \\ \tilde{g}(t) &= \hat{g}(t) * \text{sinc}\left(\tfrac{t}{\tau}\right) \end{aligned}} \quad \begin{aligned} &(6) \\ &(7) \end{aligned}$$

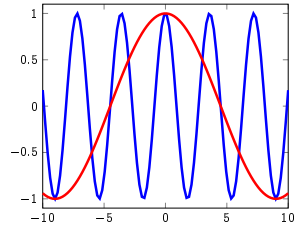
sampling theorem

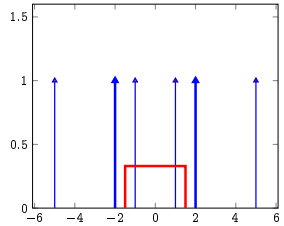
Time domain view: "convolution with a sinc" is our interpolation algorithm!

$$g(t) = \underbrace{g(t)\left[\frac{1}{\tau}\text{Ш}\left(\frac{t}{\tau}\right)\right]}_{\text{sampling}} * \underbrace{\text{sinc}\left(\frac{t}{\tau}\right)}_{\text{interpolation}} \quad \text{for } g(t) \text{ bandlimited} \tag{8}$$

aliasing

Replica (islands) overlap ⇔ not sampling fast enough



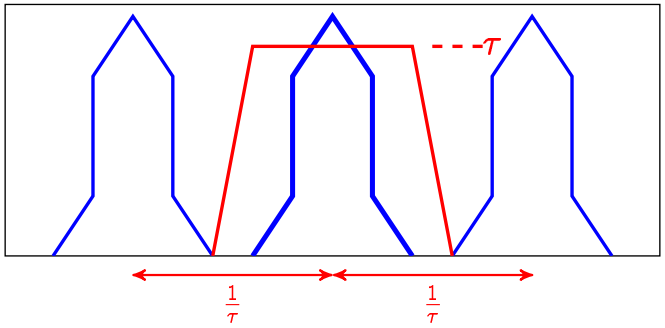


"high frequency pretending to be low frequency"

anti-aliasing: lowpass filter the original signal first

sampling

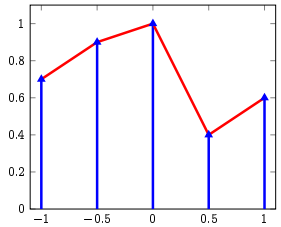
Question: Is sinc (of a particularly frequency) the only way to interpolate?



Different window can give rise to different interpolation kernel (function).

sampling

Question: If we (sample and then) perform linear interpolation, how are we shaping the signal?



This is convolution with a triangle function:

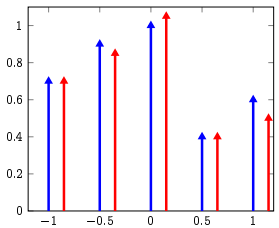
$$\tilde{g}(t) = \hat{g}(t) * \Lambda(\tfrac{t}{\tau}) \tag{9}$$

$$\tilde{G}(s) = \hat{G}(s) \cdot \tau \text{sinc}^2(\tau s) \tag{10}$$

The "$\text{sinc}^2(\tau s)$" causes a reduction of higher frequencies (compared with $\Pi(\tau s)$).

interlace sampling

Question: If we sample in a pair, how does that affect the condition for "perfect reconstruction"?



This situation is common, e.g., in capturing multiple low resolution images of an object and then perform "super-resolution".

How about having N sampling interlaced together?

interlace sampling

Somewhat surprisingly, in terms of requirement of bandlimitedness, whether the second set of sampling is at the midpoint of the first or not makes no difference!

However, if the signal has noise, then there is noise amplification in the reconstruction the closer the second set of sampling is to the first one. (Ref: E. Lam, "Noise in superresolution reconstruction," *Optics Letters*, vol. 28, no. 22, pp. 2234–2236, November 2003. doi:10.1364/OL.28.002234)

When we have N sets, where $N \to \infty$, this is Taylor's series. $\Rightarrow$ idea of analytic continuation.

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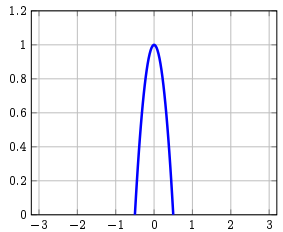
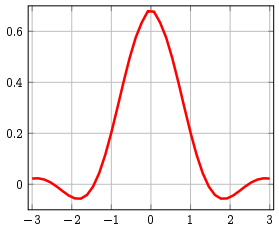
time-limited signal

Now consider a time-limited signal and we are replicating in the time domain (= sampling in the frequency domain).





In this example, $g(t) = (1 - 4t^2)\Pi(t)$. Suppose we want to replicate with a period $L = 2$.

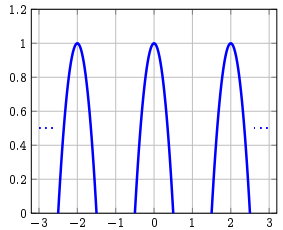
time-limited signal

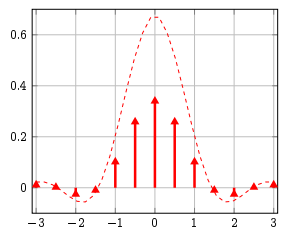
We let $\hat{g}(t)$ be the replicated signal, and $\hat{G}(s)$ its Fourier transform.

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} g(t-nL) = g(t) * \sum_{n=-\infty}^{\infty} \delta(t-nL) \tag{11}$$

$$= g(t) * \tfrac{1}{L} \sqcup\!\sqcup \left(\tfrac{t}{L}\right) \tag{12}$$

$$\hat{G}(s) = G(s) \sqcup\!\sqcup (sL) \tag{13}$$





time-limited signal

Let us consider the sampling in the frequency domain:

$$\sqcup\!\sqcup(sL) = \tfrac{1}{L} \sum_{n=-\infty}^{\infty} \delta\left(s - \tfrac{n}{L}\right) \tag{14}$$

Therefore,

$$\hat{G}(s) = G(s) \sqcup\!\sqcup (sL) \tag{15}$$

$$= \tfrac{1}{L} \sum_{n=-\infty}^{\infty} G(s)\delta\left(s - \tfrac{n}{L}\right) \tag{16}$$

$$= \sum_{n=-\infty}^{\infty} \left[\tfrac{1}{L} G\left(\tfrac{n}{L}\right)\right] \delta\left(s - \tfrac{n}{L}\right) \tag{17}$$

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} \left[\tfrac{1}{L} G\left(\tfrac{n}{L}\right)\right] e^{j2\pi \frac{n}{L} t} \tag{18}$$

time-limited signal

We now have two ways to represent $\hat{g}(t)$:

$$\boxed{\hat{g}(t) = \sum_{n=-\infty}^{\infty} g(t-nL) = \sum_{n=-\infty}^{\infty} \underbrace{\left[\tfrac{1}{L} G\left(\tfrac{n}{L}\right)\right]}_{=\alpha_n} e^{j2\pi \frac{n}{L} t}} \tag{19}$$

A periodic function is decomposed into a weighted sum of exponentials.

This is Fourier series!

$$\alpha_n = \tfrac{1}{L} G\left(\tfrac{n}{L}\right) = \tfrac{1}{L} \int_{-\infty}^{\infty} g(t) e^{-j2\pi s t} dt \Big|_{s=\frac{n}{L}} \tag{20}$$

$$\boxed{\underbrace{\alpha_n}_{\text{Fourier coefficient}} = \frac{1}{L} \int_{-L/2}^{L/2} g(t) e^{-j2\pi \frac{n}{L} t} dt} \tag{21}$$

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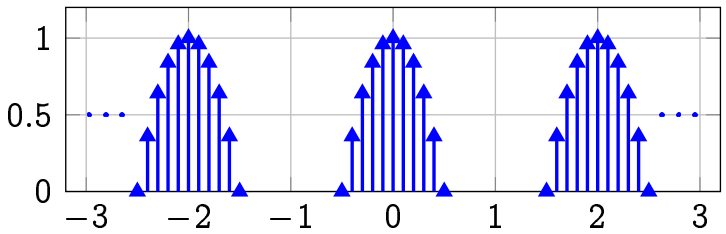
Discrete periodic signal

Suppose we have a sampled signal that is periodic.

$$f(t) = (1 - 4t^2)\Pi(t)$$
$$g(t) = f(t) \cdot \tfrac{1}{M} \sqcup (\tfrac{t}{M}) \qquad (M = 0.1)$$
$$\widehat{g}(t) = \sum_{n=-\infty}^{\infty} g(t - nL) \qquad (L = 2)$$



$$\boxed{\text{Define: } P = \tfrac{L}{M}} \tag{22}$$

Discrete periodic signal

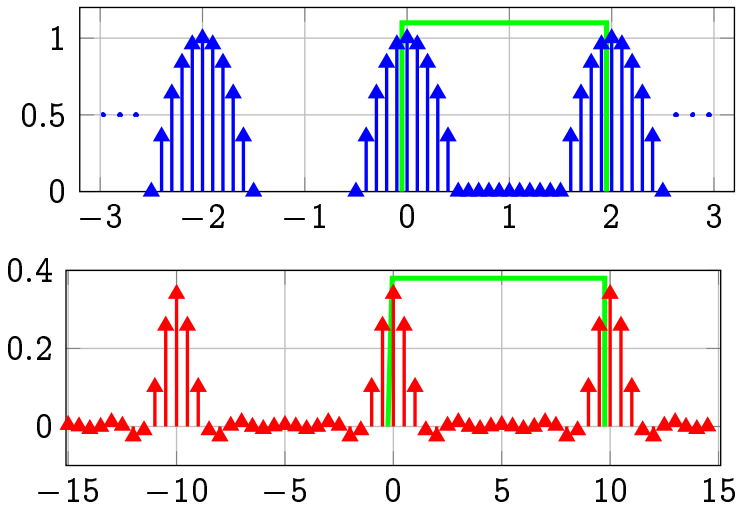
From Eq. (21),

$$\alpha_n = \frac{1}{L}\int_{-L/2}^{L/2} g(t) e^{-j2\pi \frac{n}{L} t} dt \tag{23}$$
$$= \frac{1}{L}\int_{-L/2}^{L/2} \left[\sum_{k=-L/2M}^{(L/2M)-1} f(t)\delta(t - kM)\right] e^{-j2\pi \frac{n}{L} t} dt \tag{24}$$
$$= \frac{1}{L}\sum_{k=-L/2M}^{(L/2M)-1} f(kM) e^{-j2\pi \frac{n}{L} kM} \tag{25}$$
$$= \frac{1}{L}\sum_{k=-P/2}^{(P/2)-1} \underbrace{f(kM)}_{\triangleq f[k]} e^{-j2\pi \frac{nk}{P}} \tag{26}$$

This is discrete Fourier transform (DFT)!

Discrete Fourier transform

We can relate one period (with length $P = L/M$) of the signal in both time and frequency domains:



Discrete Fourier transform

More commonly, we define DFT as follows. Suppose we have a length-N signal $g[\tau]$ with its DFT also a length-N signal $G[\nu]$.

$$g[\tau], \quad \tau = 0, 1, \ldots, N-1$$
$$G[\nu], \quad \nu = 0, 1, \ldots, N-1$$

These are the DFT equations:

$$G[\nu] = \frac{1}{N}\sum_{\tau=0}^{N-1} g[\tau] e^{-j2\pi \frac{\nu\tau}{N}} \tag{27}$$
$$g[\tau] = \sum_{\nu=0}^{N-1} G[\nu] e^{j2\pi \frac{\nu\tau}{N}} \tag{28}$$

Discrete Fourier transform

It is more straightforward to show that any length-N signal can go through a DFT and inverse DFT to get back the original signal:

$$\sum_{\nu=0}^{N-1} G[\nu] e^{j2\pi \frac{\nu\tau}{N}} = \sum_{\nu=0}^{N-1} \left(\frac{1}{N} \sum_{\tau'=0}^{N-1} g[\tau'] e^{-j2\pi \frac{\nu\tau'}{N}} \right) e^{j2\pi \frac{\nu\tau}{N}} \tag{29}$$

$$= \frac{1}{N} \sum_{\tau'=0}^{N-1} g[\tau'] \sum_{\nu=0}^{N-1} e^{-j2\pi \frac{\nu}{N}(\tau'-\tau)} \tag{30}$$

$$= \frac{1}{N} \sum_{\tau'=0}^{N-1} g[\tau'] \cdot \begin{cases} N & \text{if } \tau' = \tau \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

$$= g[\tau] \tag{32}$$