

Homework 2

Due: Mar 31, 2026, 11:55pm

Instruction: Submit your answers electronically through Moodle. In Moodle, you must submit under the PDF Submission link with ONE PDF file containing answers to all remaining questions.

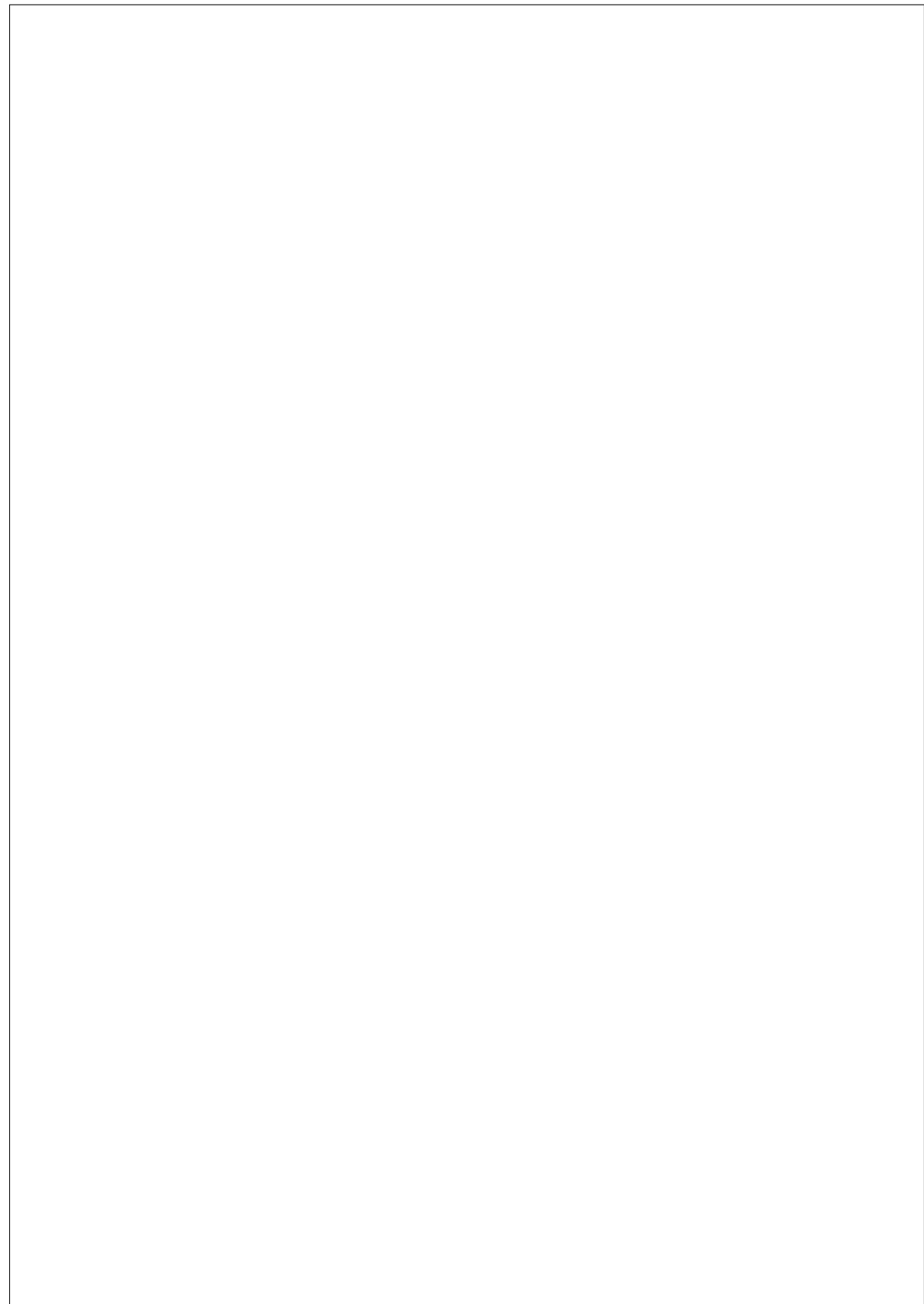
Question 1

A certain continuous random variable X has probabilities given by

$$\mathbb{P}(X > x) = \begin{cases} \left(\frac{x_m}{x}\right)^\alpha & x \geq x_m \\ 1 & x < x_m \end{cases}$$

There are two parameters:

- A scale parameter, x_m , which denotes the minimum possible value for X . Its value must be positive.
 - A shape parameter, α , which must also be positive.
1. Find the cumulative density function (CDF) of X .
 2. Find the probability density function (PDF) of X .
 3. Find the mean of X . Assume that $\alpha > 1$.
 4. Find the median of X .
 5. Find the mode of X .



Question 2

A signal processing unit analyzes the incoming measurements using a two-stage pipeline. Let

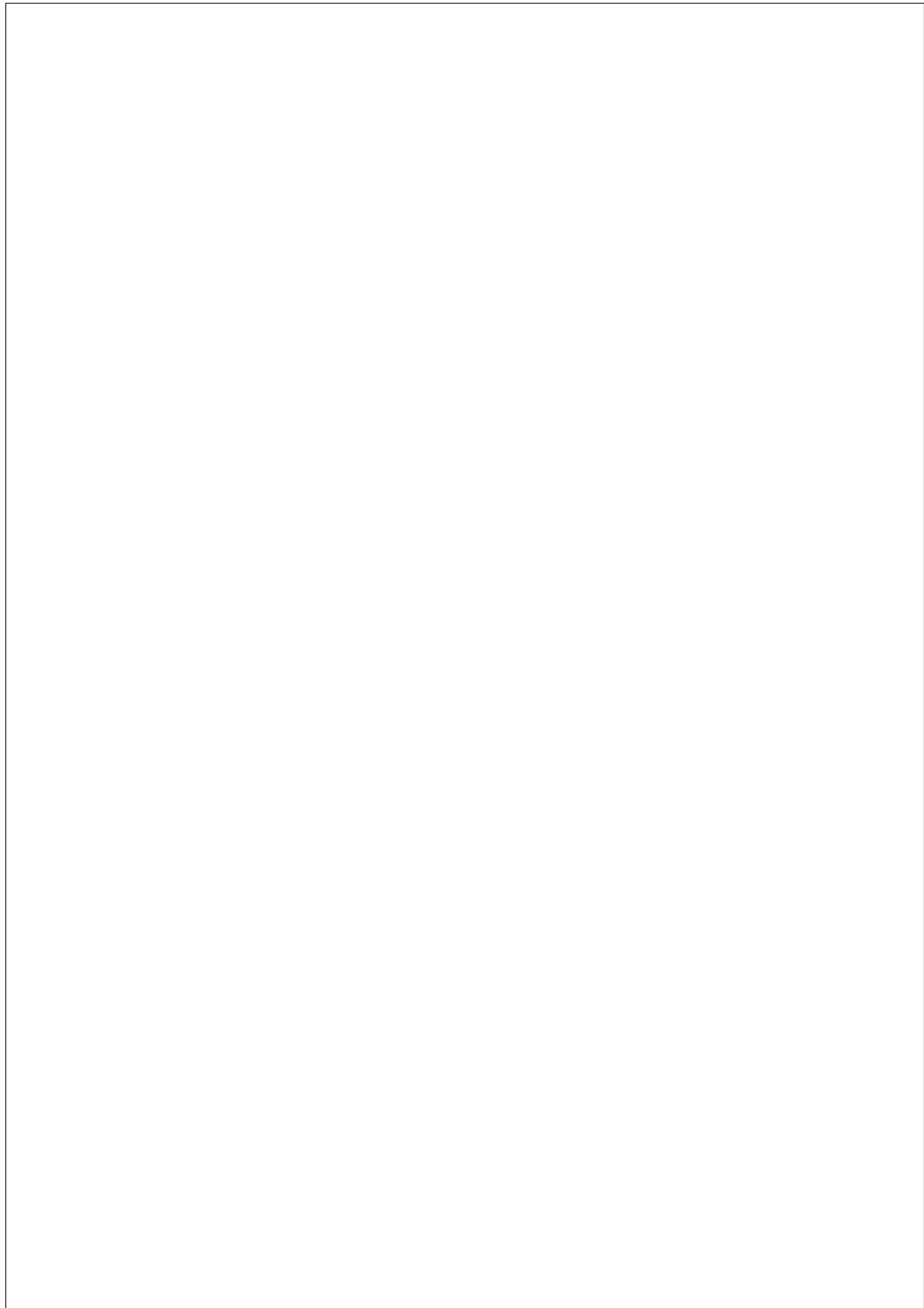
X = processing time of Stage 1, and Y = total processing time of the system.

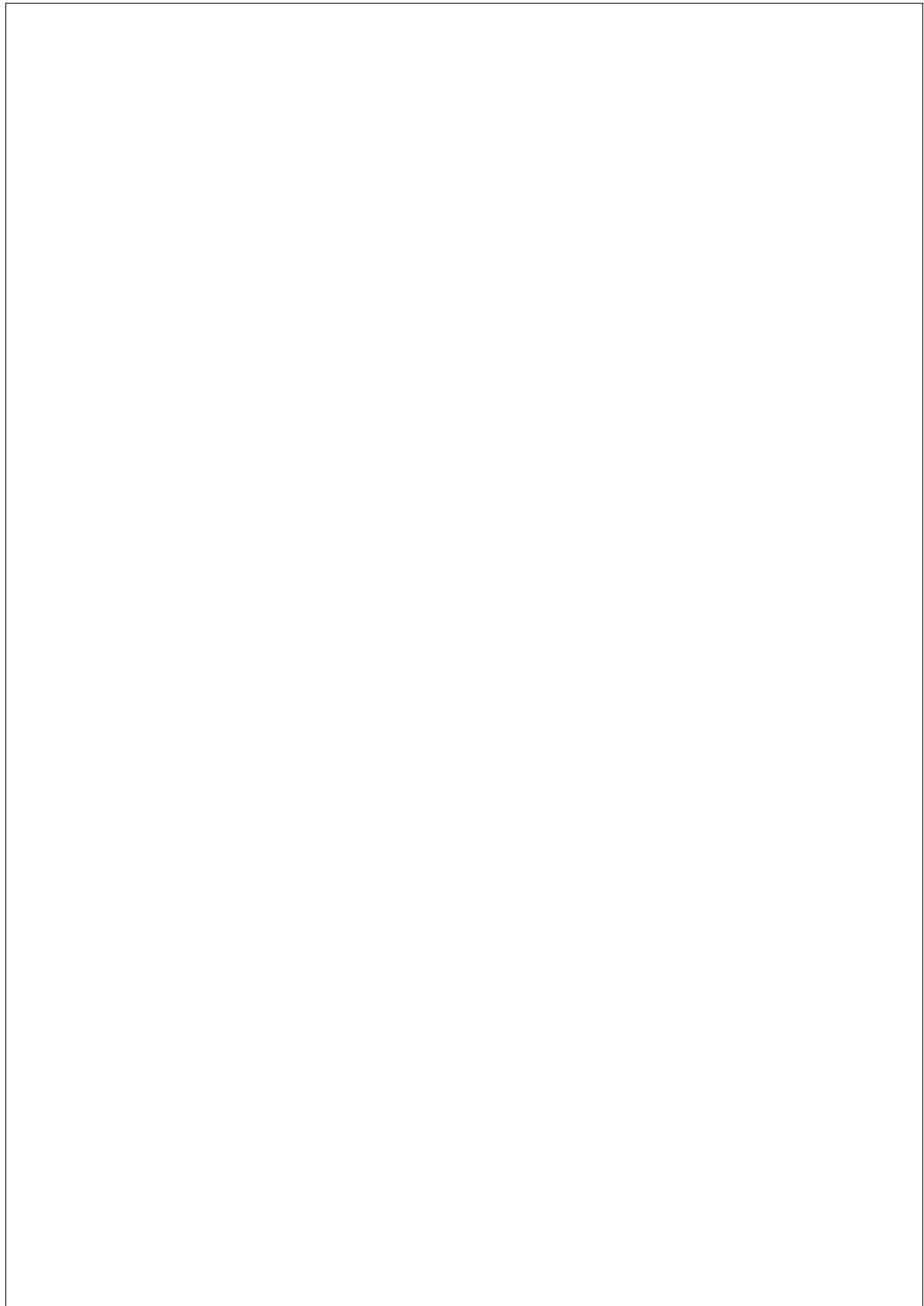
Because Stage 2 can only begin after Stage 1 has completed, the processing time satisfies $0 \leq X \leq Y$. The joint probability density function (PDF) of X and Y is given by

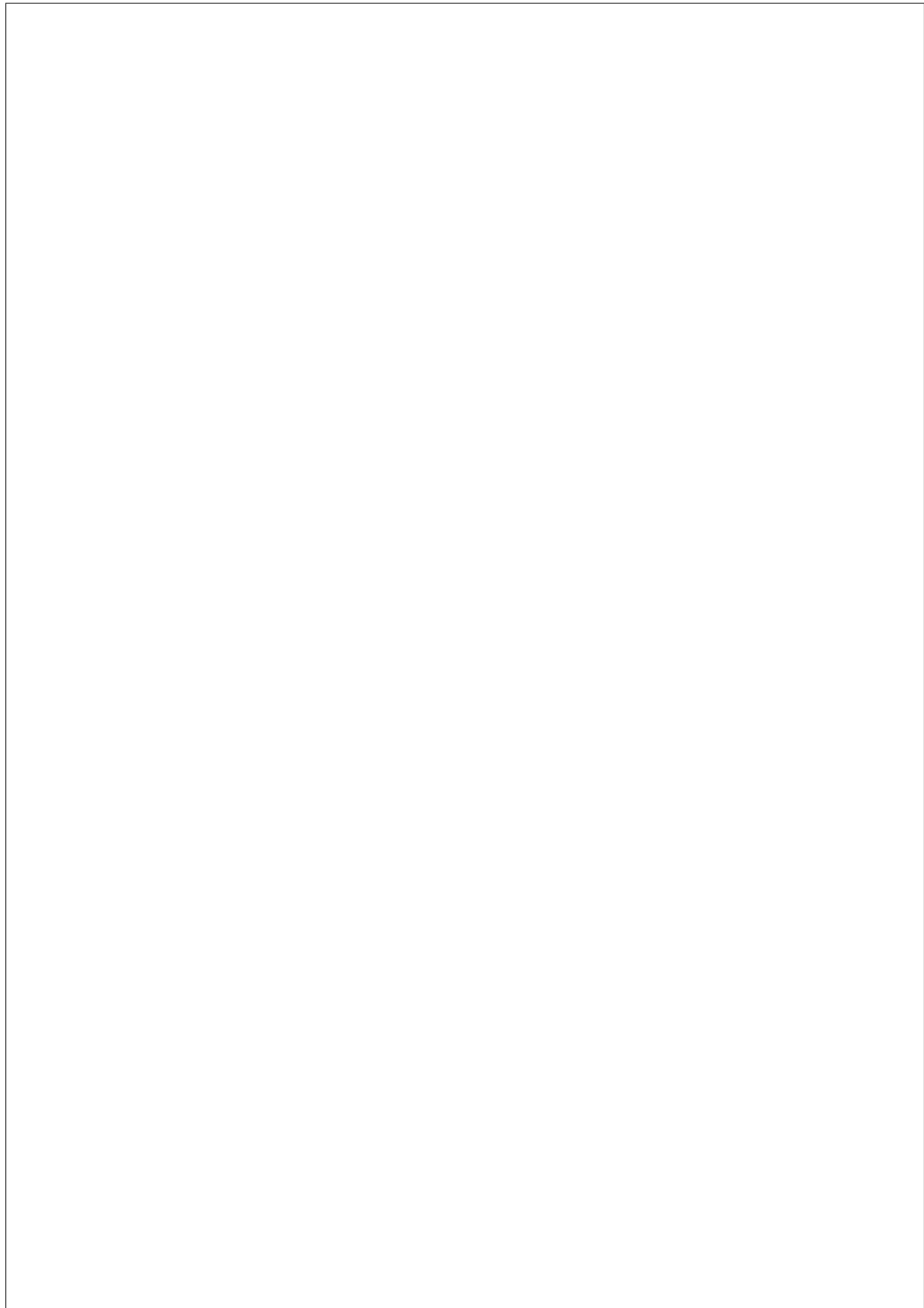
$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \leq x \leq y < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is a system parameter. All times are measured in seconds.

1. (a) Determine the marginal PDF $f_Y(y)$ and cumulative density function (CDF) $F_Y(y)$ of Y .
 (b) Compute the mean $\mathbb{E}(Y)$.
2. The system controller raises a warning if the Stage 1 processing time exceeds half of the total processing time, i.e., $X > \frac{Y}{2}$. Compute the probability that a warning is raised.
3. Define $Z = Y - X$, which represents the processing time of Stage 2.
 - (a) Find the joint pdf $f_{X,Z}(x,z)$.
 - (b) Are X and Z independent? Justify your answer.







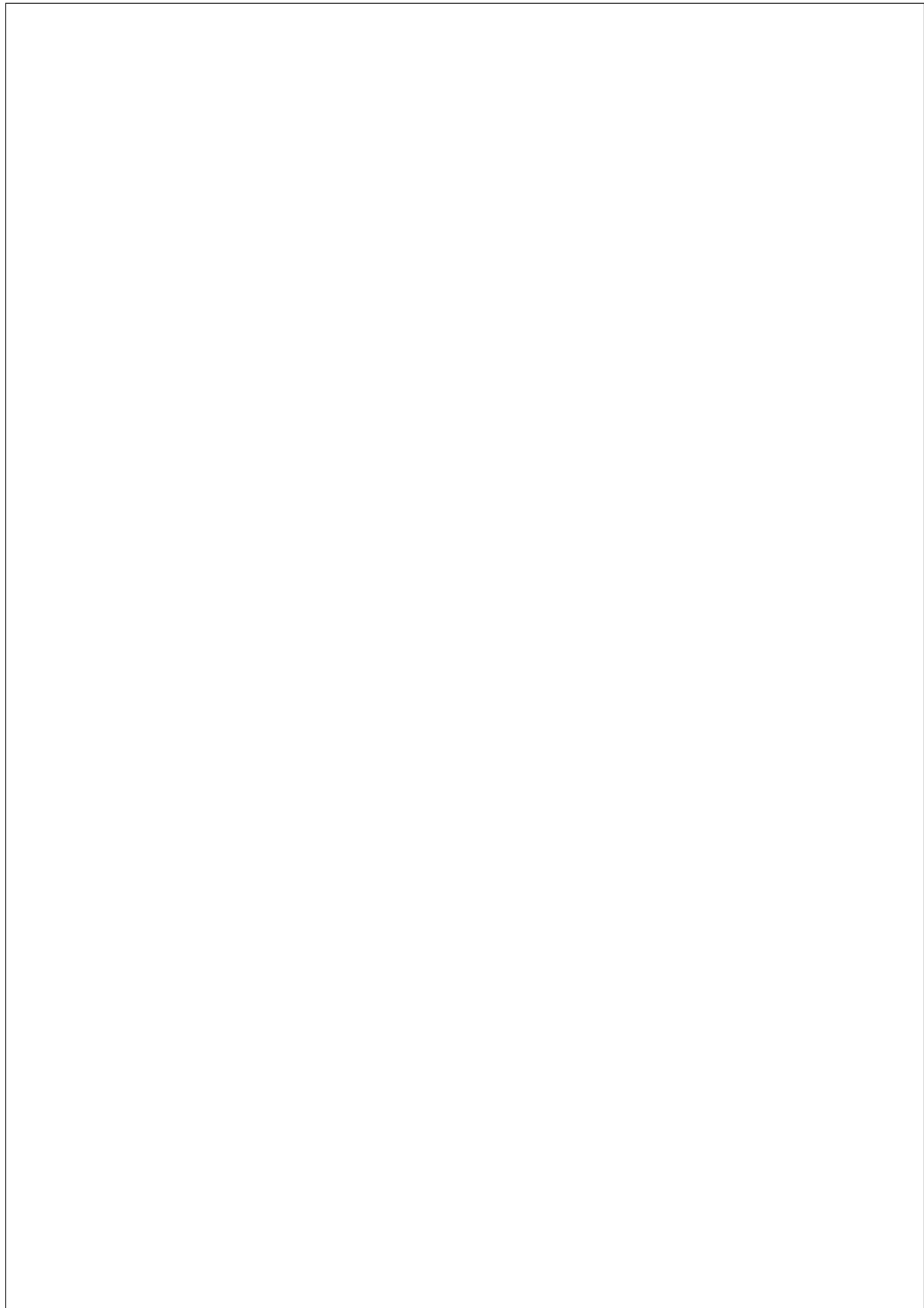
Question 3

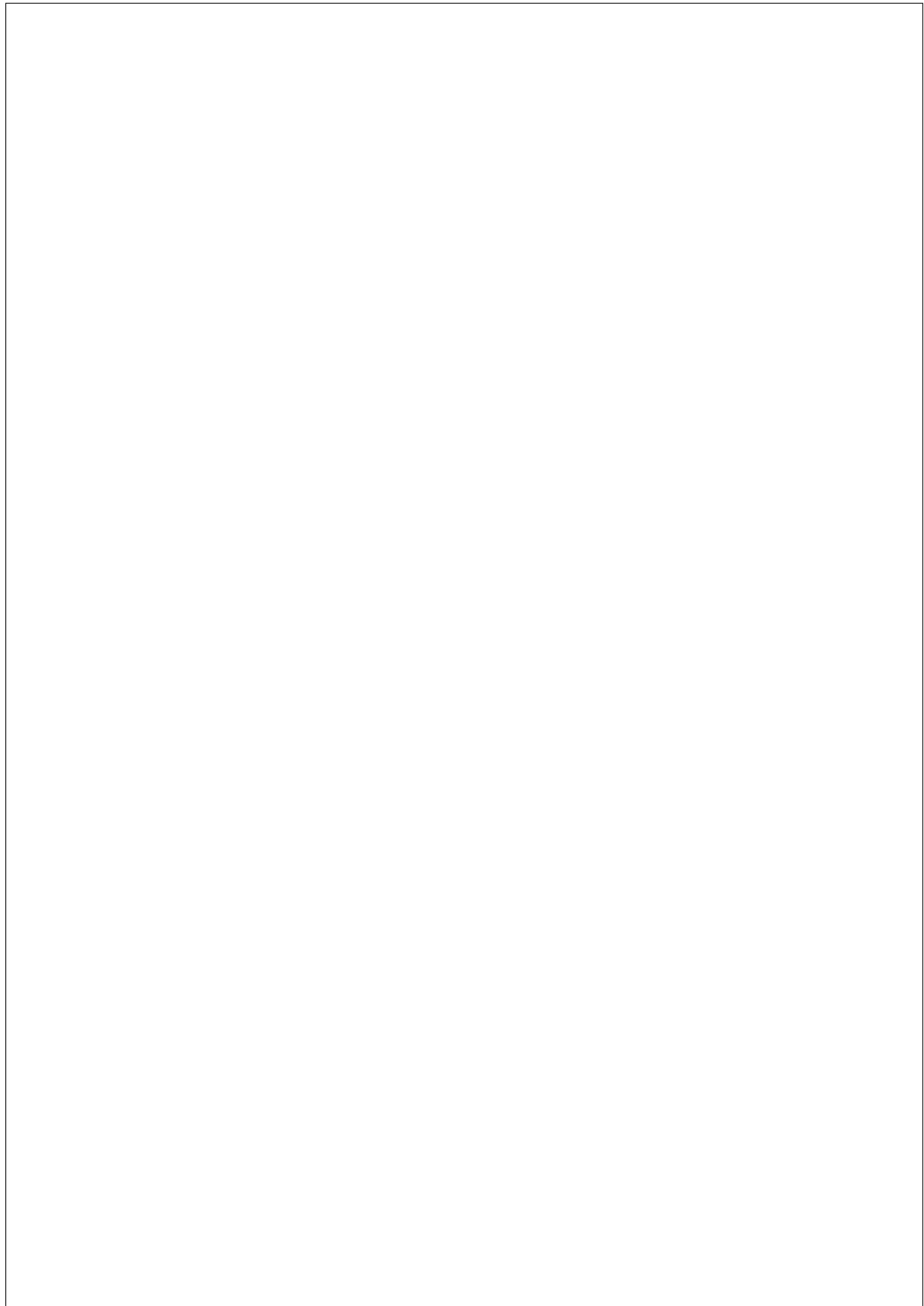
At a self-checkout kiosk (a small, standalone, often digital, self-service terminal used in high-traffic areas like malls) in the Learning Commons, the basic processing time of a transaction is a continuous random variable X , which is uniformly distributed over 0 to 2 minutes. Independently of X , a random ID verification check is triggered with probability 0.3.

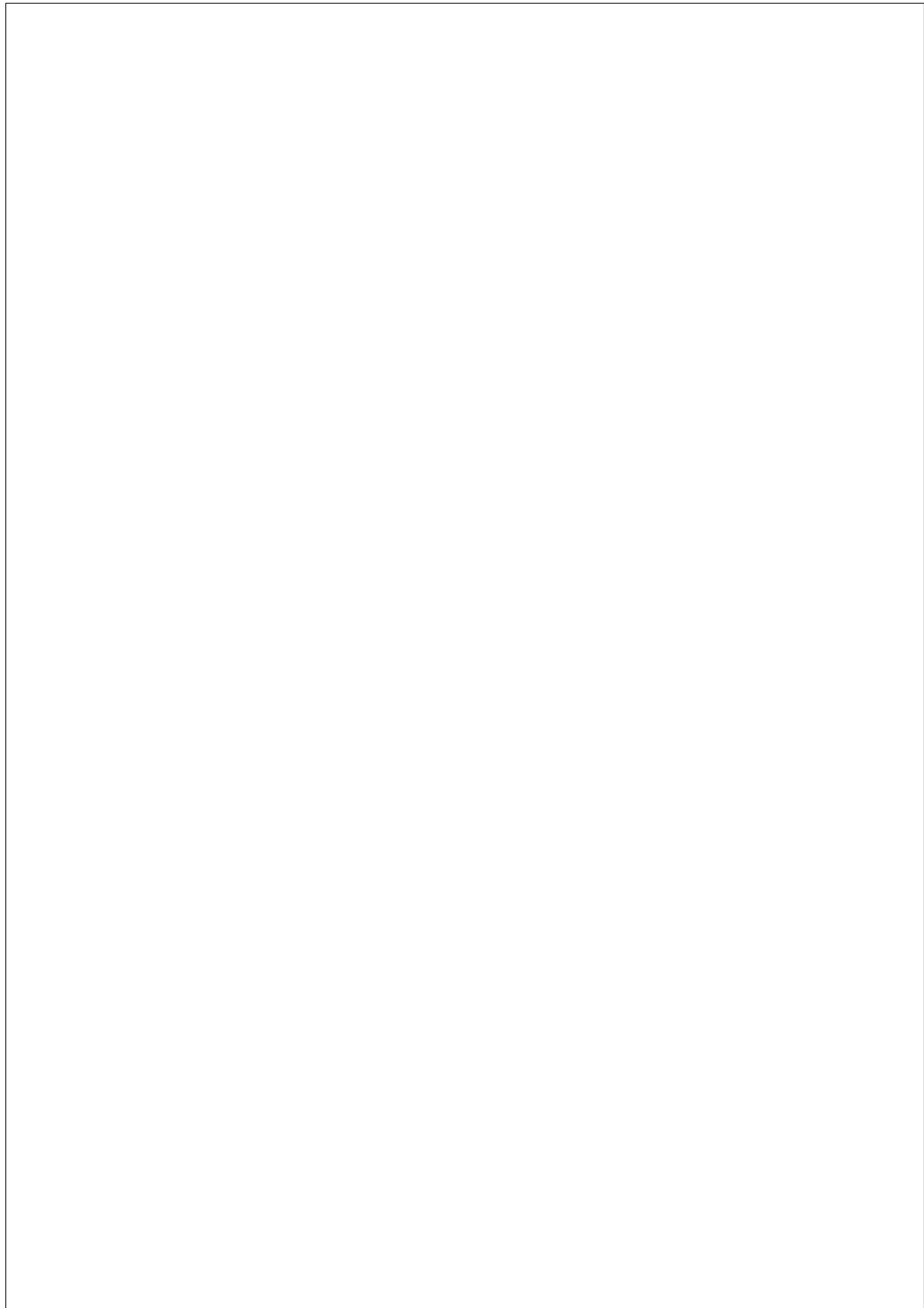
If the verification check is not triggered, then the total processing time is simply $Y = X$. If the verification check is triggered, then exactly one extra minute is added, so the total processing time is $Y = X + 1$.

Let C denote the event that the verification check is triggered.

1. Find the CDF $F_Y(y)$ and the PDF $f_Y(y)$ of Y .
2. Find $\mathbb{P}(Y \leq 1.2)$.
3. Find $\mathbb{E}(Y)$.
4. Given that the total processing time exceeds 1.5 minutes, find $\mathbb{P}(C | Y > 1.5)$.







Question 4

The lifetime of two electronic devices are independent, and modeled as exponential random variables X and Y , with parameters λ and μ , respectively. The two devices are put in parallel, so that only when both are broken, then the whole system will fail. Let $Z = \max\{X, Y\}$ be the random variable denoting the lifetime of the whole system.

1. Find the CDF of Z , i.e., $F_Z(z)$.
2. Find the PDF of Z , i.e., $f_Z(z)$.
3. Find the expected value of Z , i.e., $\mathbb{E}(Z)$.
4. Find the expected value of $\min\{X, Y\}$, i.e., the expected time when either device fails.
(*Hint: You can use $X + Y = \max\{X, Y\} + \min\{X, Y\}$ and take expected values on both sides, without the need to derive the distribution of $\min\{X, Y\}$.)*)

Question 5

A particle is placed at the origin of a two-dimensional coordinate system. The particle moves once per second according to one of the following rules.

1. *Discrete Direction Case:* In each step, the particle moves one unit of distance. The direction is equally likely to be up, down, left, or right (each with probability $1/4$). After n seconds, draw a circle with the particle's distance from the origin as the radius. Find the expected value of the area of this circle.
2. *Continuous Direction Case:* In each step, the particle moves one unit of distance. The direction angle θ with the x -axis is uniformly distributed over $(0, 2\pi)$.
 - (a) After n seconds, draw a circle with the particle's distance from the origin as the radius. Find the expected value of the area of this circle.
 - (b) Suppose the distance moved in each step is not fixed at 1, but is a random variable $L_k > 0$. Assume $L_1, L_2, \dots, L_n$ are independent and identically distributed, following an exponential distribution with parameter λ . The direction angle θ is still uniformly distributed over $(0, 2\pi)$ and is independent of the step length. After n seconds, draw a circle with the particle's distance from the origin as the radius. Find the expected value of the area of this circle.

