

Homework 2

Due: Mar 31, 2026, 11:55pm

Instruction: Submit your answers electronically through Moodle. In Moodle, you must submit under the PDF Submission link with ONE PDF file containing answers to all remaining questions.

Question 1

A certain continuous random variable X has probabilities given by

$$\mathbb{P}(X > x) = \begin{cases} \left(\frac{x_m}{x}\right)^\alpha & x \geq x_m \\ 1 & x < x_m \end{cases}$$

There are two parameters:

- A scale parameter, x_m , which denotes the minimum possible value for X . Its value must be positive.
 - A shape parameter, α , which must also be positive.
1. Find the cumulative density function (CDF) of X .
 2. Find the probability density function (PDF) of X .
 3. Find the mean of X . Assume that $\alpha > 1$.
 4. Find the median of X .
 5. Find the mode of X .

1. We can find the CDF directly from the given expression:

$$F_X(x) = 1 - \mathbb{P}(X > x) = \begin{cases} 1 - \left(\frac{x_m}{x}\right)^\alpha & x \geq x_m \\ 0 & x < x_m \end{cases}$$

2. We can find the PDF by differentiation:

$$f_X(x) = \begin{cases} \frac{\alpha \cdot x_m^\alpha}{x^{\alpha+1}} & x \geq x_m \\ 0 & x < x_m \end{cases}$$

3. To find the mean, we evaluate the integral:

$$\begin{aligned}
 \mathbb{E}(X) &= \int_{x_m}^{\infty} x \cdot \frac{\alpha \cdot x_m^{\alpha}}{x^{\alpha+1}} dx \\
 &= \alpha \cdot x_m^{\alpha} \int_{x_m}^{\infty} x^{-\alpha} dx \\
 &= \alpha \cdot x_m^{\alpha} \left[\frac{1}{-\alpha+1} x^{-\alpha+1} \right]_{x_m}^{\infty} \\
 &= \alpha \cdot x_m^{\alpha} \left(0 - \frac{x_m^{-\alpha+1}}{-\alpha+1} \right) \\
 &= \frac{\alpha}{\alpha-1} x_m
 \end{aligned}$$

Alternatively, we can also find it by:

$$\begin{aligned}
 \mathbb{E}(X) &= \int_0^{\infty} (1 - F_X(x)) dx \\
 &= \int_0^{x_m} (1) dx + \int_{x_m}^{\infty} \left(\frac{x_m}{x} \right)^{\alpha} dx \\
 &= x_m + x_m^{\alpha} \left(0 - \frac{x_m^{-\alpha+1}}{-\alpha+1} \right) \\
 &= x_m + \frac{x_m}{\alpha-1} \\
 &= \frac{\alpha}{\alpha-1} x_m
 \end{aligned}$$

4. To find the median, we set the CDF to $\frac{1}{2}$.

$$\begin{aligned}
 1 - \left(\frac{x_m}{x} \right)^{\alpha} &= \frac{1}{2} \\
 \frac{x_m}{x} &= \left(\frac{1}{2} \right)^{1/\alpha} \\
 x &= 2^{1/\alpha} \cdot x_m
 \end{aligned}$$

5. To find the mode, we note that the PDF is strictly decreasing for $x \geq x_m$. Hence, the maximum value is at $x = x_m$.

Question 2

A signal processing unit analyzes the incoming measurements using a two-stage pipeline. Let

X = processing time of Stage 1, and Y = total processing time of the system.

Because Stage 2 can only begin after Stage 1 has completed, the processing time satisfies $0 \leq X \leq Y$. The joint probability density function (PDF) of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \leq x \leq y < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is a system parameter. All times are measured in seconds.

1. (a) Determine the marginal PDF $f_Y(y)$ and cumulative density function (CDF) $F_Y(y)$ of Y .
(b) Compute the mean $\mathbb{E}(Y)$.
2. The system controller raises a warning if the Stage 1 processing time exceeds half of the total processing time, i.e., $X > \frac{Y}{2}$. Compute the probability that a warning is raised.
3. Define $Z = Y - X$, which represents the processing time of Stage 2.
 - (a) Find the joint pdf $f_{X,Z}(x,z)$.
 - (b) Are X and Z independent? Justify your answer.

1. (a) The marginal density is obtained by integrating the joint density over the variable x . Since it ranges from 0 to y , we have

$$f_Y(y) = \int_0^y f_{X,Y}(x, y) dx = \int_0^y \lambda^2 e^{-\lambda y} dx = \lambda^2 y e^{-\lambda y}, \quad y \geq 0,$$

because the integrand does not depend on x .

Next, the CDF of Y is given by

$$F_Y(y) = \begin{cases} \int_0^y \lambda^2 t e^{-\lambda t} dt & y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

To compute this integral, we apply integration by parts. Letting

$$u = t, \quad dv = \lambda^2 e^{-\lambda t} dt,$$

we then have

$$du = dt, \quad v = -\lambda e^{-\lambda t}.$$

Hence, using integration by parts,

$$\begin{aligned} \int_0^y \lambda^2 t e^{-\lambda t} dt &= [-\lambda t e^{-\lambda t}]_0^y + \int_0^y \lambda e^{-\lambda t} dt \\ &= -\lambda y e^{-\lambda y} + [-e^{-\lambda t}]_0^y \\ &= -\lambda y e^{-\lambda y} - e^{-\lambda y} + 1 \\ &= 1 - e^{-\lambda y} (1 + \lambda y). \end{aligned}$$

1. (b) We can make use of the formula

$$\mathbb{E}(Y) = \int_0^{\infty} (1 - F_Y(y)) \, dy = \int_0^{\infty} (e^{-\lambda y} + \lambda y e^{-\lambda y}) \, dy$$

We need to evaluate $\int_0^{\infty} y e^{-\lambda y} \, dy$, which we make use of integration by parts with

$$u = y, \quad dv = e^{-\lambda y} \, dy, \quad du = dy, \quad v = -\frac{1}{\lambda} e^{-\lambda y}.$$

Then,

$$\begin{aligned} \int_0^{\infty} y e^{-\lambda y} \, dy &= \left[-\frac{y}{\lambda} e^{-\lambda y} \right]_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda y} \, dy \\ &= 0 + \frac{1}{\lambda} \left[-\frac{1}{\lambda} e^{-\lambda y} \right]_0^{\infty} \\ &= \frac{1}{\lambda^2}. \end{aligned}$$

Substituting back to the first formula,

$$\begin{aligned} \mathbb{E}(Y) &= \int_0^{\infty} e^{-\lambda y} \, dy + \int_0^{\infty} \lambda y e^{-\lambda y} \, dy \\ &= \frac{1}{\lambda} + \lambda \cdot \frac{1}{\lambda^2} \\ &= \frac{2}{\lambda} \end{aligned}$$

Alternatively, it is possible to compute the mean directly from the PDF $f_Y(y) = \lambda^2 y e^{-\lambda y}$, where $y > 0$. To find the mean,

$$\mathbb{E}(Y) = \int_0^{\infty} y f_Y(y) \, dy = \int_0^{\infty} y \lambda^2 y e^{-\lambda y} \, dy = \lambda^2 \int_0^{\infty} y^2 e^{-\lambda y} \, dy.$$

We compute $\int_0^{\infty} y^2 e^{-\lambda y} \, dy$ using integration by parts with

$$u = y^2, \quad dv = e^{-\lambda y} \, dy, \quad \Rightarrow \quad du = 2y \, dy, \quad v = -\frac{1}{\lambda} e^{-\lambda y}.$$

Thus,

$$\int_0^{\infty} y^2 e^{-\lambda y} \, dy = \left[-\frac{y^2}{\lambda} e^{-\lambda y} \right]_0^{\infty} + \frac{2}{\lambda} \int_0^{\infty} y e^{-\lambda y} \, dy = \frac{2}{\lambda^3}$$

Substituting back, we have

$$\mathbb{E}(Y) = \lambda^2 \cdot \frac{2}{\lambda^3} = \frac{2}{\lambda}$$

2. We want to compute the probability $\mathbb{P}(X > \frac{Y}{2})$. The region where this event occurs satisfies the inequalities $0 \leq x \leq y$ and $x > \frac{y}{2}$. Therefore, we can express the probability as the double integral

$$\mathbb{P}\left(X > \frac{Y}{2}\right) = \int_0^\infty \int_{y/2}^y \lambda^2 e^{-\lambda y} dx dy.$$

Since the integrand does not depend on x , the inner integral over x simplifies to

$$\int_{y/2}^y \lambda^2 e^{-\lambda y} dx = \lambda^2 e^{-\lambda y} \left(y - \frac{y}{2}\right) = \frac{\lambda^2 y}{2} e^{-\lambda y}.$$

Substituting this back, the probability becomes

$$\mathbb{P}\left(X > \frac{Y}{2}\right) = \frac{\lambda^2}{2} \int_0^\infty y e^{-\lambda y} dy.$$

Using the integration by parts we computed for the previous part, where we obtained $\int_0^\infty y e^{-\lambda y} dy = \frac{1}{\lambda^2}$, we now have

$$\mathbb{P}\left(X > \frac{Y}{2}\right) = \frac{1}{2}.$$

3. (a) We have $Y = X + Z$. Therefore, the joint PDF of X and Z can be expressed as

$$f_{X,Z}(x, z) = f_{X,Y}(x, x + z).$$

Substituting the given joint density, we have

$$f_{X,Z}(x, z) = \lambda^2 e^{-\lambda(x+z)}, \quad x \geq 0, \quad z \geq 0.$$

- (b) The marginal PDF of X is found by integrating over z :

$$f_X(x) = \int_0^\infty \lambda^2 e^{-\lambda(x+z)} dz = \lambda^2 e^{-\lambda x} \int_0^\infty e^{-\lambda z} dz = \lambda^2 e^{-\lambda x} \left(\frac{1}{\lambda}\right) = \lambda e^{-\lambda x}.$$

Similarly, the marginal PDF of Z is

$$f_Z(z) = \lambda e^{-\lambda z}.$$

Since $f_X(x) f_Z(z) = f_{X,Z}(x, z)$, it follows that the variables X and Z are independent.

Question 3

At a self-checkout kiosk (a small, standalone, often digital, self-service terminal used in high-traffic areas like malls) in the Learning Commons, the basic processing time of a transaction is a continuous random variable X , which is uniformly distributed over 0 to 2 minutes. Independently of X , a random ID verification check is triggered with probability 0.3.

If the verification check is not triggered, then the total processing time is simply $Y = X$. If the verification check is triggered, then exactly one extra minute is added, so the total processing time is $Y = X + 1$.

Let C denote the event that the verification check is triggered.

1. Find the CDF $F_Y(y)$ and the PDF $f_Y(y)$ of Y .
2. Find $\mathbb{P}(Y \leq 1.2)$.
3. Find $\mathbb{E}(Y)$.
4. Given that the total processing time exceeds 1.5 minutes, find $\mathbb{P}(C | Y > 1.5)$.

From the question, we know $\mathbb{P}(C) = 0.3$ and $\mathbb{P}(C^c) = 0.7$.
Also, since $X \sim U(0, 2)$, its CDF is

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{2}, & 0 \leq x \leq 2, \\ 1, & x > 2. \end{cases}$$

1. For any real number y , conditioning on whether the verification check is triggered,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y \leq y | C^c) \mathbb{P}(C^c) + \mathbb{P}(Y \leq y | C) \mathbb{P}(C).$$

Under C^c , we have $Y = X$, and under C , we have $Y = X + 1$. Hence,

$$F_Y(y) = 0.7 \mathbb{P}(X \leq y) + 0.3 \mathbb{P}(X + 1 \leq y) = 0.7F_X(y) + 0.3F_X(y - 1).$$

Now consider different ranges of y .

For $y < 0$:

$$F_Y(y) = 0.$$

For $0 \leq y < 1$: we have $F_X(y) = y/2$ and $F_X(y - 1) = 0$, so

$$F_Y(y) = 0.7 \cdot \frac{y}{2} = 0.35y.$$

For $1 \leq y < 2$: we have $F_X(y) = y/2$ and $F_X(y - 1) = (y - 1)/2$, so

$$F_Y(y) = 0.7 \cdot \frac{y}{2} + 0.3 \cdot \frac{y - 1}{2} = 0.5y - 0.15.$$

For $2 \leq y \leq 3$: we have $F_X(y) = 1$ and $F_X(y - 1) = (y - 1)/2$, so

$$F_Y(y) = 0.7 \cdot 1 + 0.3 \cdot \frac{y - 1}{2} = 0.55 + 0.15y.$$

For $y > 3$:

$$F_Y(y) = 1.$$

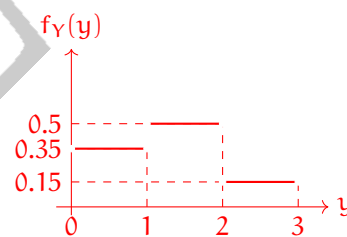
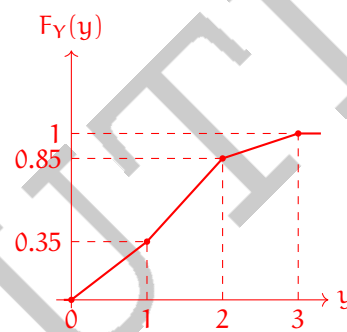
Therefore,

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ 0.35y, & 0 \leq y < 1, \\ 0.5y - 0.15, & 1 \leq y < 2, \\ 0.55 + 0.15y, & 2 \leq y \leq 3, \\ 1, & y > 3. \end{cases}$$

1. Continuing from part (1), differentiating piecewise, we obtain the PDF

$$f_Y(y) = \begin{cases} 0.35, & 0 < y < 1, \\ 0.5, & 1 < y < 2, \\ 0.15, & 2 < y < 3, \\ 0, & \text{otherwise.} \end{cases}$$

The following sketches make the piecewise form of $F_Y(y)$ and $f_Y(y)$ more transparent.



2. Since $1.2 \in [1, 2)$, we use the corresponding expression for $F_Y(y)$:

$$\mathbb{P}(Y \leq 1.2) = F_Y(1.2) = \frac{1.2}{2} - 0.15 = 0.45.$$

3. Conditioning on whether the verification check is triggered,

$$\begin{aligned}\mathbb{E}(Y) &= \mathbb{E}(Y | C^c) \mathbb{P}(C^c) + \mathbb{E}(Y | C) \mathbb{P}(C) \\ &= \mathbb{E}(X) \mathbb{P}(C^c) + \mathbb{E}(X + 1) \mathbb{P}(C).\end{aligned}$$

Since $X \sim U(0, 2)$, we have $\mathbb{E}(X) = 1$. Therefore,

$$\mathbb{E}(Y) = 1 \times 0.7 + 2 \times 0.3 = 1.3.$$

4. By Bayes' rule,

$$\mathbb{P}(C | Y > 1.5) = \frac{\mathbb{P}(C \cap \{Y > 1.5\})}{\mathbb{P}(Y > 1.5)}.$$

First, if C occurs, then $Y = X + 1$. Hence,

$$\mathbb{P}(C \cap \{Y > 1.5\}) = \mathbb{P}(C) \mathbb{P}(X + 1 > 1.5) = 0.3 \mathbb{P}(X > 0.5).$$

Since $X \sim U(0, 2)$, $\mathbb{P}(X > 0.5) = \frac{2 - 0.5}{2} = \frac{3}{4}$. Thus,

$$\mathbb{P}(C \cap \{Y > 1.5\}) = 0.3 \times \frac{3}{4} = 0.225.$$

Next,

$$\begin{aligned}\mathbb{P}(Y > 1.5) &= \mathbb{P}(C^c) \mathbb{P}(X > 1.5) + \mathbb{P}(C) \mathbb{P}(X + 1 > 1.5) \\ &= 0.7 \mathbb{P}(X > 1.5) + 0.3 \mathbb{P}(X > 0.5).\end{aligned}$$

Again using the uniform distribution, $\mathbb{P}(X > 1.5) = \frac{2 - 1.5}{2} = \frac{1}{4}$. Therefore,

$$\mathbb{P}(Y > 1.5) = 0.7 \times \frac{1}{4} + 0.3 \times \frac{3}{4} = 0.175 + 0.225 = 0.4.$$

Hence, $\mathbb{P}(C | Y > 1.5) = \frac{0.225}{0.4} = \frac{9}{16} = 0.5625$.

Question 4

The lifetime of two electronic devices are independent, and modeled as exponential random variables X and Y , with parameters λ and μ , respectively. The two devices are put in parallel, so that only when both are broken, then the whole system will fail. Let $Z = \max\{X, Y\}$ be the random variable denoting the lifetime of the whole system.

1. Find the CDF of Z , i.e., $F_Z(z)$.
2. Find the PDF of Z , i.e., $f_Z(z)$.
3. Find the expected value of Z , i.e., $\mathbb{E}(Z)$.
4. Find the expected value of $\min\{X, Y\}$, i.e., the expected time when either device fails.
(Hint: You can use $X + Y = \max\{X, Y\} + \min\{X, Y\}$ and take expected values on both sides, without the need to derive the distribution of $\min\{X, Y\}$.)

1. Note that X and Y are independent, and therefore

$$\begin{aligned} F_Z(z) &= \mathbb{P}(\max\{X, Y\} \leq z) \\ &= \mathbb{P}(X \leq z) \mathbb{P}(Y \leq z) \\ &= (1 - e^{-\lambda z}) (1 - e^{-\mu z}) \end{aligned}$$

2. We can expand the expression above to get

$$F_Z(z) = 1 - e^{-\lambda z} - e^{-\mu z} + e^{-(\lambda+\mu)z}$$

Taking the derivative,

$$f_Z(z) = \lambda e^{-\lambda z} + \mu e^{-\mu z} - (\lambda + \mu) e^{-(\lambda+\mu)z}$$

3. Each part of the above is an exponential distribution, and therefore we know the expected value of the first part is $1/\lambda$, etc. Hence,

$$\mathbb{E}(Z) = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda + \mu}$$

4. Because $X + Y = \max\{X, Y\} + \min\{X, Y\}$,

$$\begin{aligned} \mathbb{E}(\min\{X, Y\}) &= \mathbb{E}(X) + \mathbb{E}(Y) - \mathbb{E}(\max\{X, Y\}) \\ &= \frac{1}{\lambda} + \frac{1}{\mu} - \left(\frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda + \mu} \right) \\ &= \frac{1}{\lambda + \mu} \end{aligned}$$

Question 5

A particle is placed at the origin of a two-dimensional coordinate system. The particle moves once per second according to one of the following rules.

1. *Discrete Direction Case:* In each step, the particle moves one unit of distance. The direction is equally likely to be up, down, left, or right (each with probability $1/4$). After n seconds, draw a circle with the particle's distance from the origin as the radius. Find the expected value of the area of this circle.
2. *Continuous Direction Case:* In each step, the particle moves one unit of distance. The direction angle θ with the x -axis is uniformly distributed over $(0, 2\pi)$.
 - (a) After n seconds, draw a circle with the particle's distance from the origin as the radius. Find the expected value of the area of this circle.
 - (b) Suppose the distance moved in each step is not fixed at 1, but is a random variable $L_k > 0$. Assume $L_1, L_2, \dots, L_n$ are independent and identically distributed, following an exponential distribution with parameter λ . The direction angle θ is still uniformly distributed over $(0, 2\pi)$ and is independent of the step length. After n seconds, draw a circle with the particle's distance from the origin as the radius. Find the expected value of the area of this circle.

1. Let X_k and Y_k denote the changes in the x- and y-coordinates in the k-th step.

Since the particle moves one unit in one of the four directions (up, down, left, or right) with equal probability,

$$(X_k, Y_k) = \begin{cases} (0, 1) \\ (0, -1) \\ (-1, 0) \\ (1, 0) \end{cases}$$

each with probability $1/4$. Hence, we have $\mathbb{E}(X_k) = 0$ and $\mathbb{E}(Y_k) = 0$.

After n seconds, the position of the particle is $\left(\sum_{k=1}^n X_k, \sum_{k=1}^n Y_k \right)$.

Let R_n denote the distance from the origin. Using Pythagoras' Theorem,

$$\begin{aligned} R_n^2 &= \left(\sum_{k=1}^n X_k \right)^2 + \left(\sum_{k=1}^n Y_k \right)^2 \\ &= \sum_{k=1}^n (X_k^2 + Y_k^2) + 2 \underbrace{\sum_{1 \leq i < j \leq n} (X_i X_j + Y_i Y_j)}_{\text{cross terms}}. \end{aligned}$$

Since the steps are independent, the expectations of the cross terms are zero, because $\mathbb{E}(X_i X_j) = \mathbb{E}(X_i) \mathbb{E}(X_j) = 0$ and $\mathbb{E}(Y_i Y_j) = \mathbb{E}(Y_i) \mathbb{E}(Y_j) = 0$, for $i \neq j$. Furthermore, $X_k^2 + Y_k^2 = 1$. Therefore, $\mathbb{E}(R_n^2) = n$.

If A denotes the area of the circle, then $A = \pi R_n^2$. Hence,

$$\mathbb{E}(A) = \pi \mathbb{E}(R_n^2) = n\pi.$$

2. (a) Let θ_k be the direction angle in the k -th step. The random variables $\theta_1, \theta_2, \dots, \theta_n$ are independent and each is uniformly distributed over $(0, 2\pi)$.

In each step, $X_k = \cos \theta_k$ and $Y_k = \sin \theta_k$. Hence, $\mathbb{E}(X_k) = 0$ and $\mathbb{E}(Y_k) = 0$.

Same as the previous part, after n seconds, the position is $\left(\sum_{k=1}^n X_k, \sum_{k=1}^n Y_k \right)$. Therefore,

$$R_n^2 = \left(\sum_{k=1}^n X_k \right)^2 + \left(\sum_{k=1}^n Y_k \right)^2.$$

Again, expanding and taking expectation, the cross terms vanish.

Moreover, $X_k^2 + Y_k^2 = \cos^2 \theta_k + \sin^2 \theta_k = 1$. Hence, $\mathbb{E}(R_n^2) = n$, and therefore,

$$\mathbb{E}(A) = \pi \mathbb{E}(R_n^2) = n\pi.$$

- (b) We now have $X_k = L_k \cos \theta_k$ and $Y_k = L_k \sin \theta_k$. Also, $X_k^2 + Y_k^2 = L_k^2$.

Proceeding in a way similar to the above,

$$\begin{aligned} R_n^2 &= \left(\sum_{k=1}^n X_k \right)^2 + \left(\sum_{k=1}^n Y_k \right)^2 \\ \mathbb{E}(R_n^2) &= \mathbb{E} \left(\sum_{k=1}^n (X_k^2 + Y_k^2) \right) + \mathbb{E} \left(2 \sum_{1 \leq i < j \leq n} (X_i X_j + Y_i Y_j) \right) \\ &= \mathbb{E} \left(\sum_{k=1}^n L_k^2 \right) \\ &= n \mathbb{E}(L^2). \end{aligned}$$

For $L \sim \text{Exp}(\lambda)$, we know that $\mathbb{E}(L) = \frac{1}{\lambda}$ and $\text{var}(L) = \frac{1}{\lambda^2}$. Therefore,

$$\mathbb{E}(L^2) = \text{var}(L) + (\mathbb{E}(L))^2 = \frac{2}{\lambda^2}.$$

Hence, $\mathbb{E}(R_n^2) = \frac{2n}{\lambda^2}$, and therefore,

$$\mathbb{E}(A) = \pi \mathbb{E}(R_n^2) = \frac{2n\pi}{\lambda^2}.$$