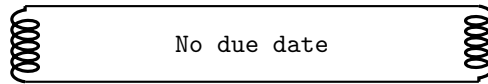


Exercise 2



Instruction: These are for your own practice. There is no need to submit your answer.

Question 1

Let $f_X(x) = \theta x + \frac{1}{2}$ for $-1 \leq x \leq 1$ and zero otherwise, and that θ is a constant.

1. For what value(s) of θ is $f_X(x)$ a valid PDF?
2. For that/those value(s) of θ , find $\mathbb{E}(X)$ and $\text{var}(X)$.
3. For what value(s) of θ is $\text{var}(X)$ maximized?

Question 2

A certain object is present with equal likelihood anywhere within a circular region of radius r . Let X be the distance of this object from the center of the region.

1. Find the CDF of X .
2. Find the PDF of X .
3. Find $\mathbb{E}(X)$.
4. Find $\text{var}(X)$.

Question 3

The continuous random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2} & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

1. Find the value of c .
2. Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.
3. Let $Y = X^2$. Calculate the conditional expectation and conditional variance of Y given A .

Question 4

The joint PDF of random variables X and Y is

$$f(x, y) = \begin{cases} Ce^{-(2x+4y)} & x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

1. Compute C .
2. Compute the joint CDF $F_{X,Y}(x, y)$.
3. Compute $\mathbb{P}(Y \leq X)$.
4. Compute the conditional PDF $f_{Y|X}(y|x)$.

Question 5

Suppose X is a continuous uniform random variable between 0 and 1, i.e., $X \sim \mathcal{U}(0, 1)$. Find the PDF of $Y = e^X$.

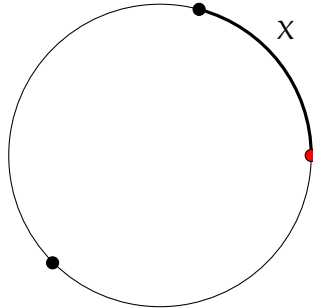
Question 6

We have three independent exponential random variables X , Y , and W , all with parameters λ .

1. What is the PDF of $Z = X + Y$?
2. What is the PDF of $Z = X + Y + W$?

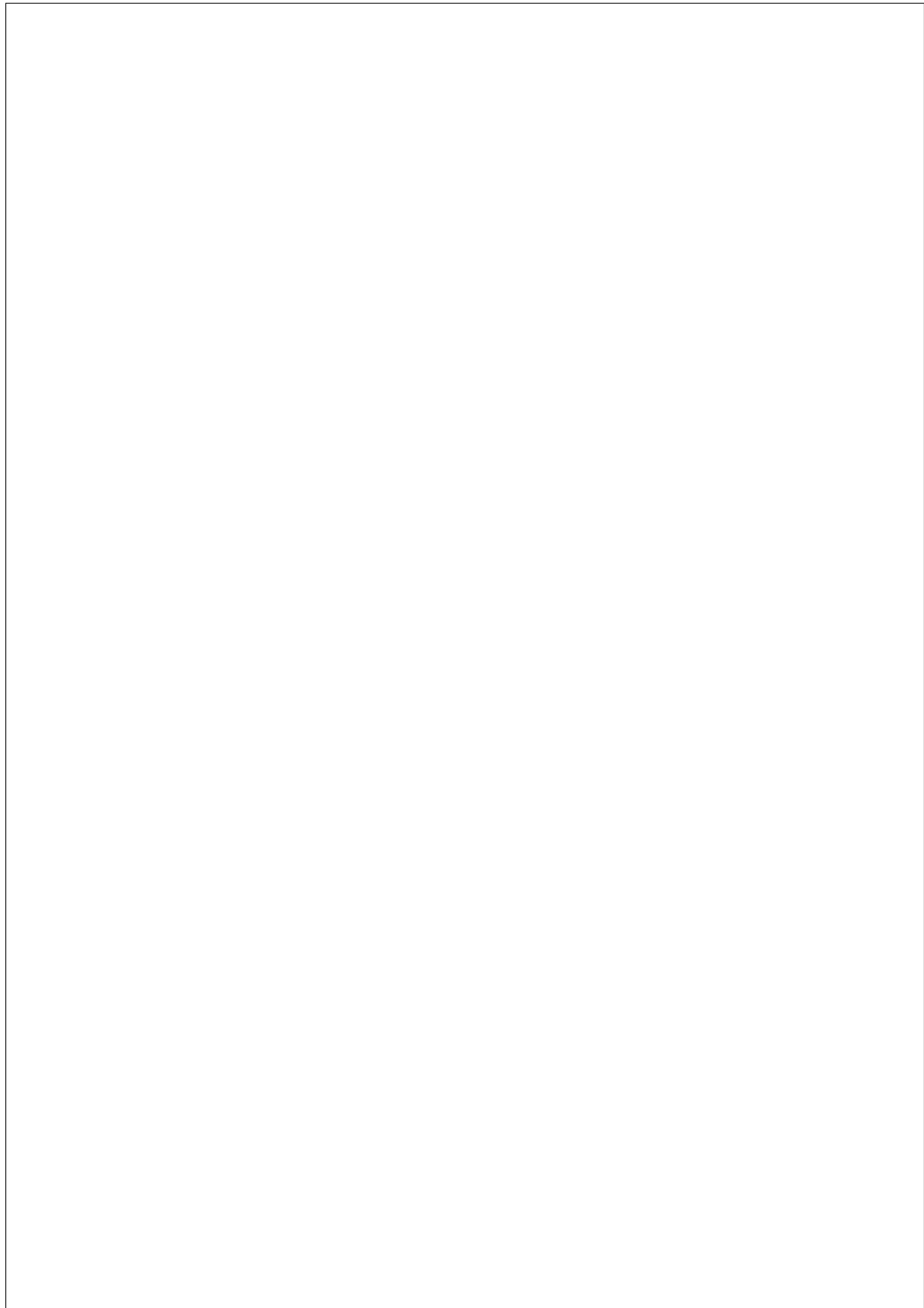
Question 7

Assume three points, one of which is red, are uniformly and independently distributed over the circumference of a circular track, which has a total length of 1. Equivalently, you can interpret the circle as a unit interval with its two ends tied together, and all distances of points are measured in terms of the length on the track. The Figure below gives a simple illustration.



1. Let X be the distance of the red point to its neighbor in the counter-clockwise direction. Find the PDF of X .
2. Let Y be the minimum distance of the red point to its nearest neighbor (either in the clockwise or counter-clockwise direction, whichever is smaller). Find the PDF of Y .
3. Let Z be the minimum distance between any two of the points. Find the PDF of Z .

Hint: You can represent the event $\{Z > z\}$ in terms of two random variables that are uniformly distributed on a unit square. To find $\mathbb{P}(Z > z)$, determine the area that corresponds to $\{Z > z\}$ graphically.



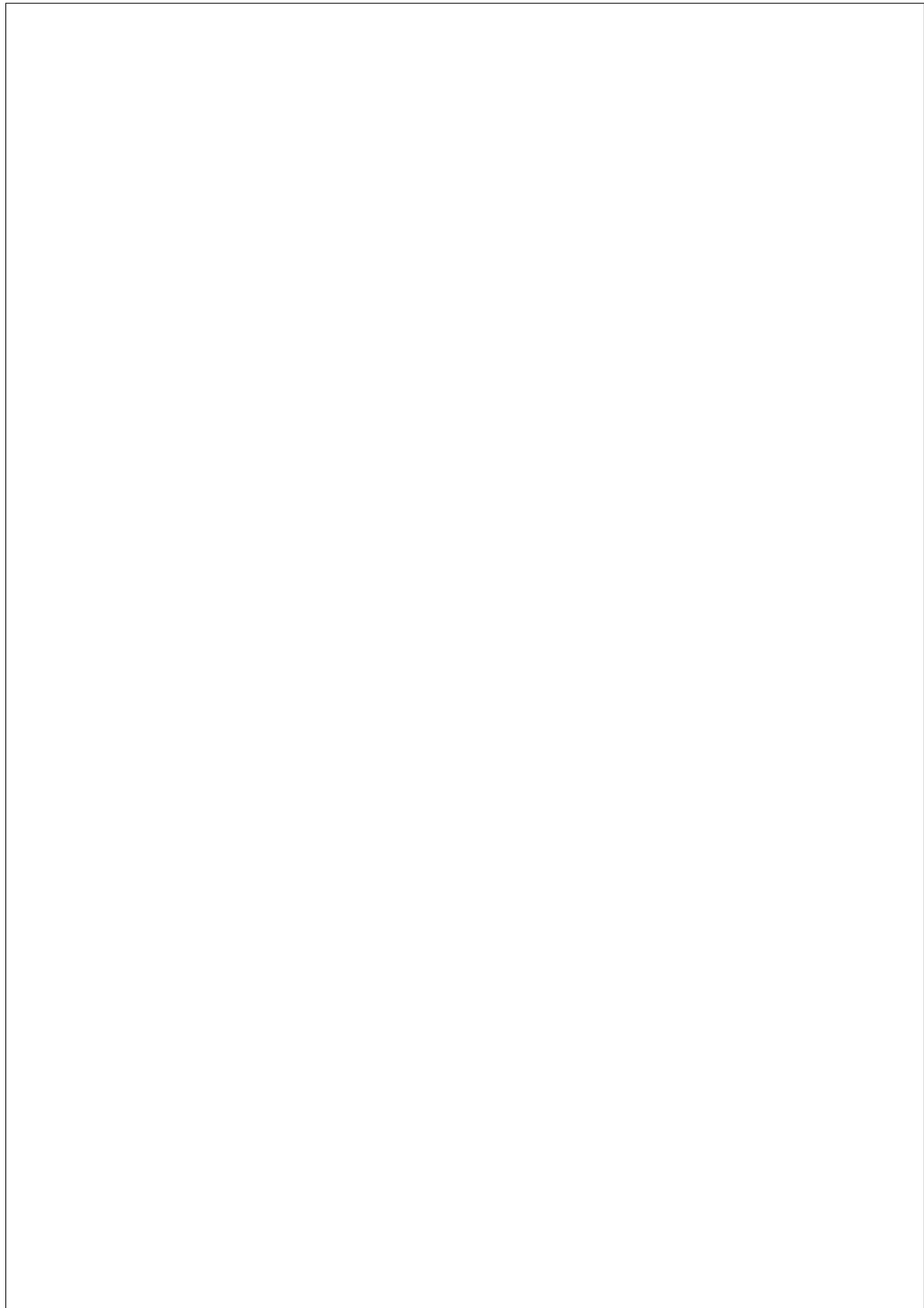
Question 8

The PDF of the Rayleigh random variable is given by

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find:

1. the CDF;
2. the mean;
3. the median;
4. the mode.



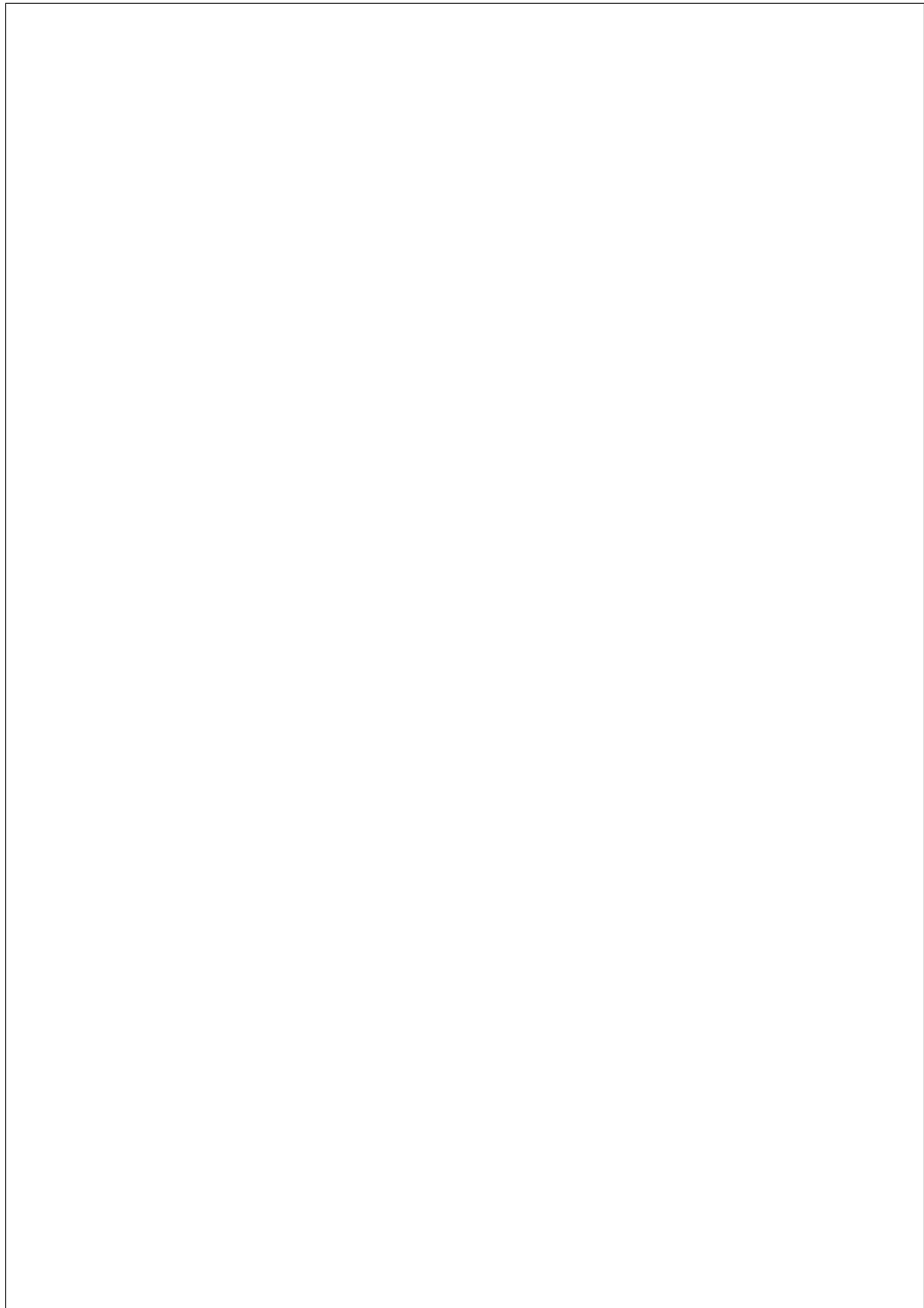
Question 9

A dairy farm produces two different types of milk. Let X = the amount (in liter) of type A on hand and Y = the amount of type B on hand. Suppose the joint PDF of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} kxy & x \geq 0, \quad y \geq 0, \quad 20 \leq x+y \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

where k is a certain constant to be determined.

1. Draw the region of positive density, and determine the value of k .
2. Are X and Y independent variables? Derive the marginal PDF of each variable.
3. Compute $\mathbb{P}(X+Y) \leq 25$.
4. What is the expected total amount of milk on hand? (correct to one decimal place)



Question 10

A dart lands uniformly random on a circular target of radius r . Let x and y be the Cartesian coordinates. Thus, we can view x to be a realization of a random variable X , and y a realization of another random variable Y .

1. Are X and Y independent?
2. Are X and Y uncorrelated?

Question 11

A dart lands uniformly random within a rhombus-shaped target bounded by the following four equations, where x and y are the Cartesian coordinates:

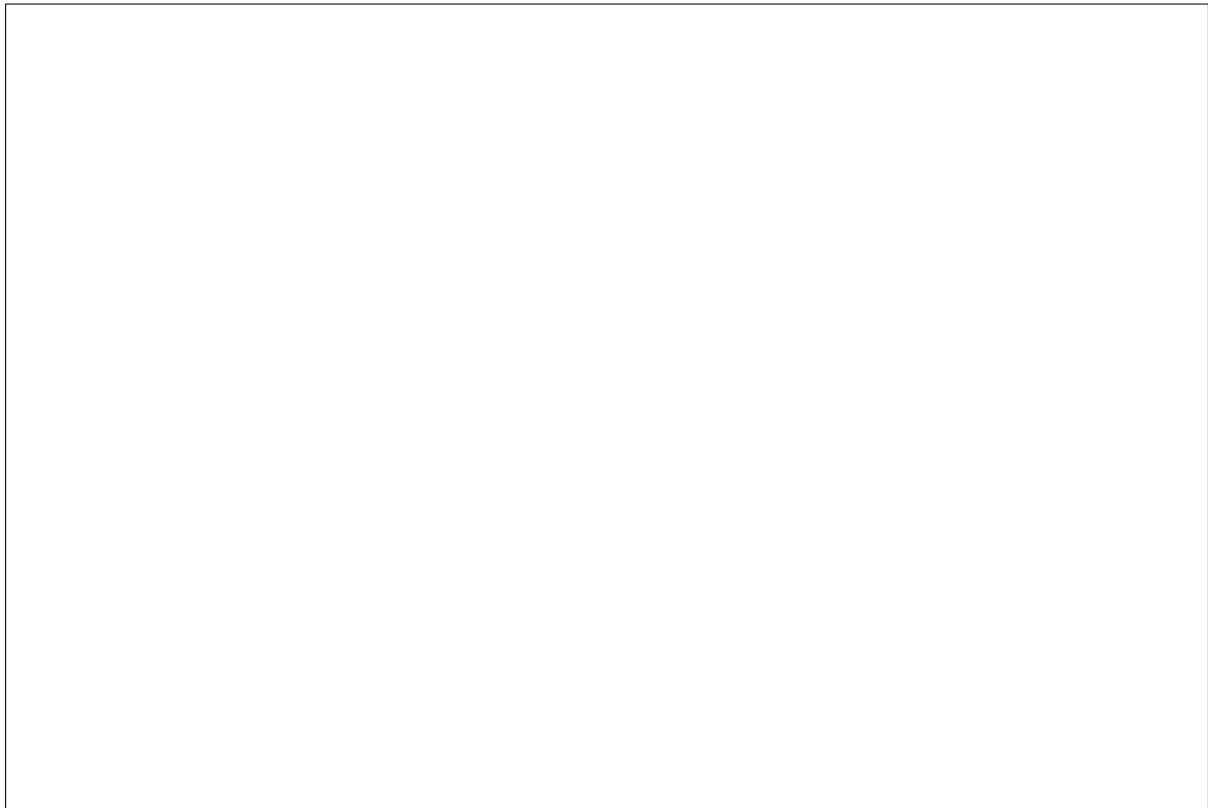
$$x + 2y = 2 \quad \text{for } x \geq 0, y \geq 0$$

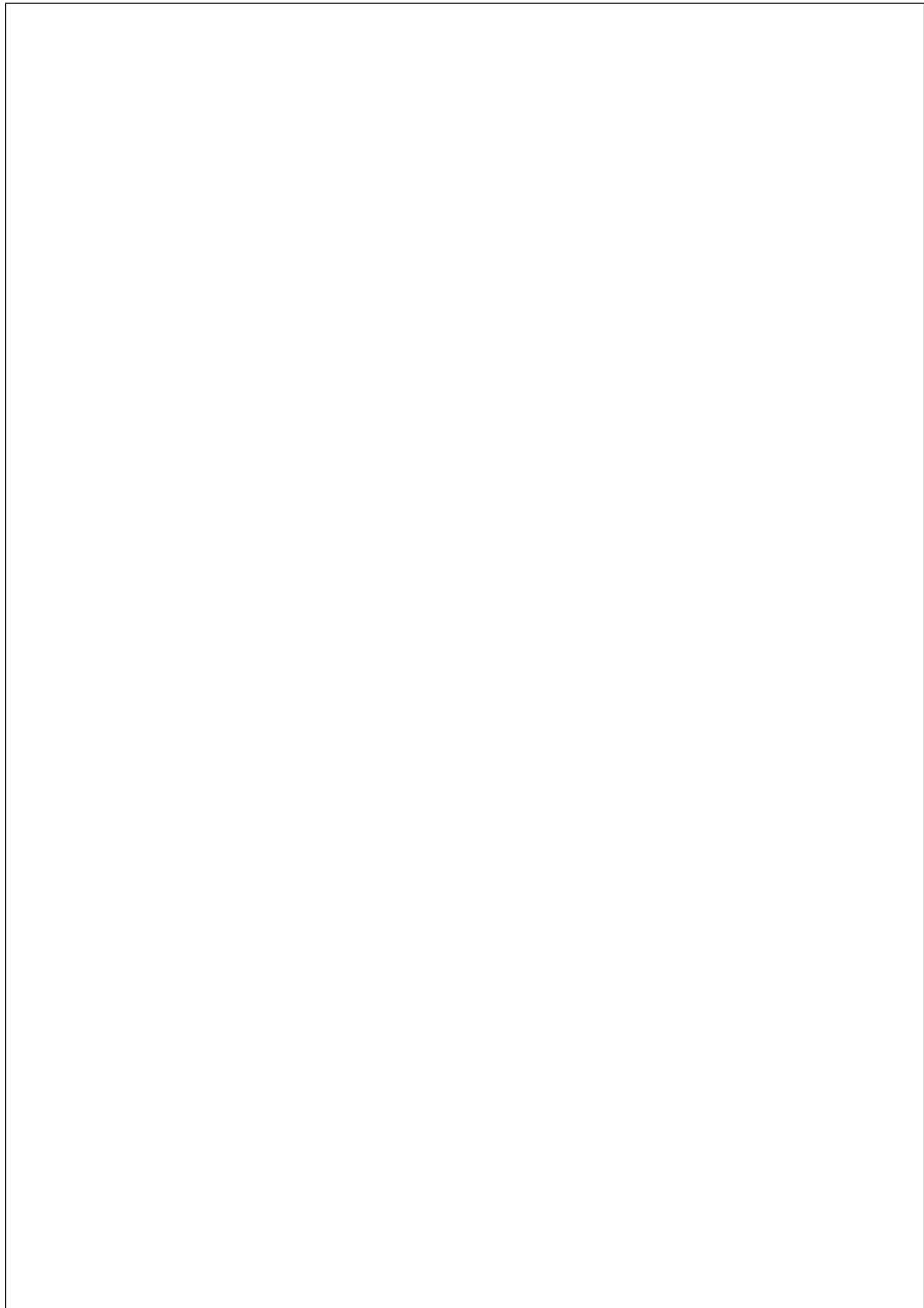
$$x - 2y = -2 \quad \text{for } x < 0, y \geq 0$$

$$x + 2y = -2 \quad \text{for } x < 0, y < 0$$

$$x - 2y = 2 \quad \text{for } x \geq 0, y < 0$$

1. Sketch the shape of the target. Mark clearly the coordinates of the corners.
2. Find the marginal PDF $f_Y(y)$.
3. Find the conditional PDF $f_{X|Y}(x|y)$.





Question 12

Alex, an avid hiker, visits Wildflower Park, renowned for its breathtaking scenery and abundant wildflowers. In one of the open areas, Alex not only counts the number of wildflowers he observes but also records the time (in minutes) he spends at the spot. Let the discrete random variable X denote the number of wildflowers observed by Alex on a single visit, with possible values 0, 1, 2, and 3 having the probability mass function:

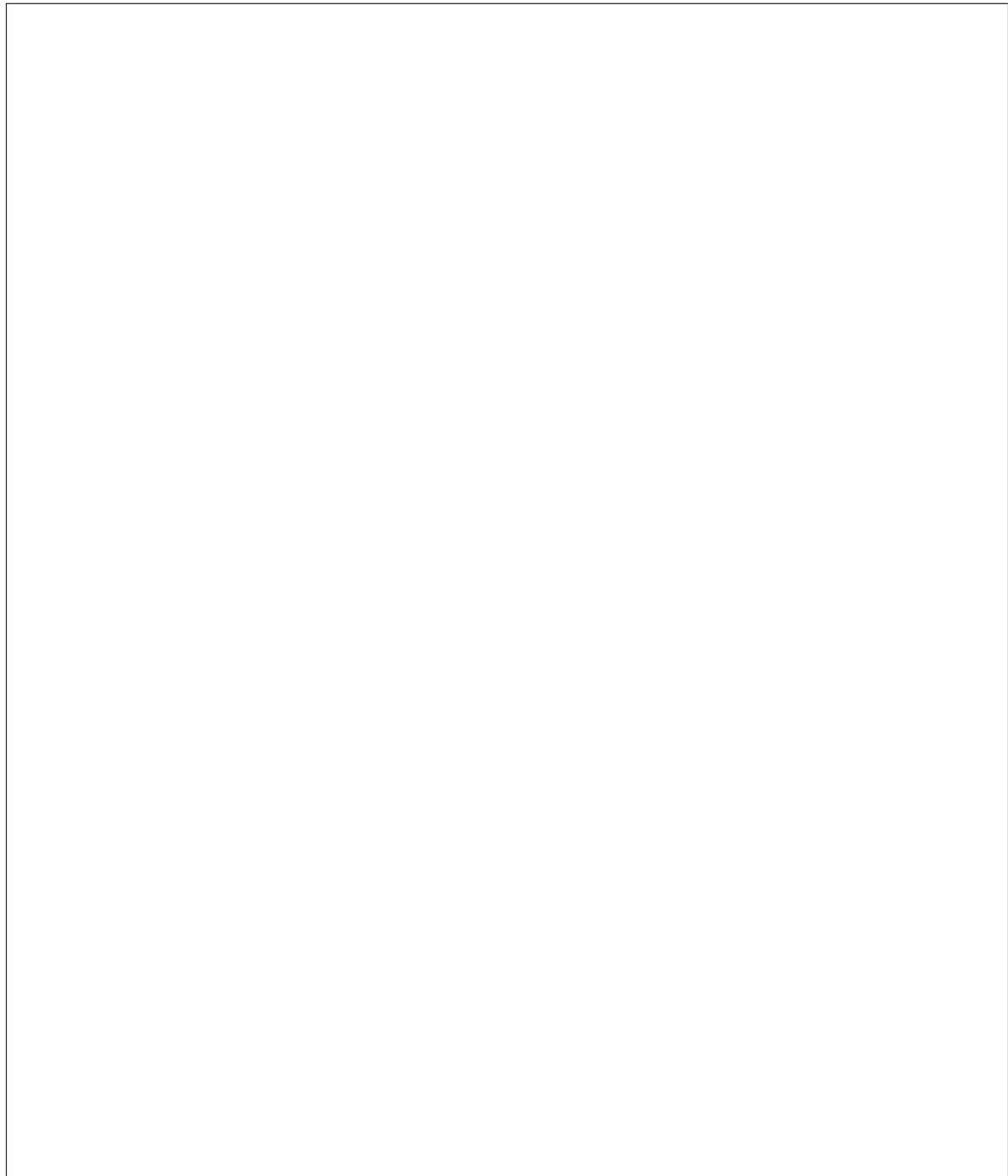
$$\mathbb{P}(X = 0) = 0.1, \quad \mathbb{P}(X = 1) = 0.4, \quad \mathbb{P}(X = 2) = 0.3, \quad \mathbb{P}(X = 3) = 0.2.$$

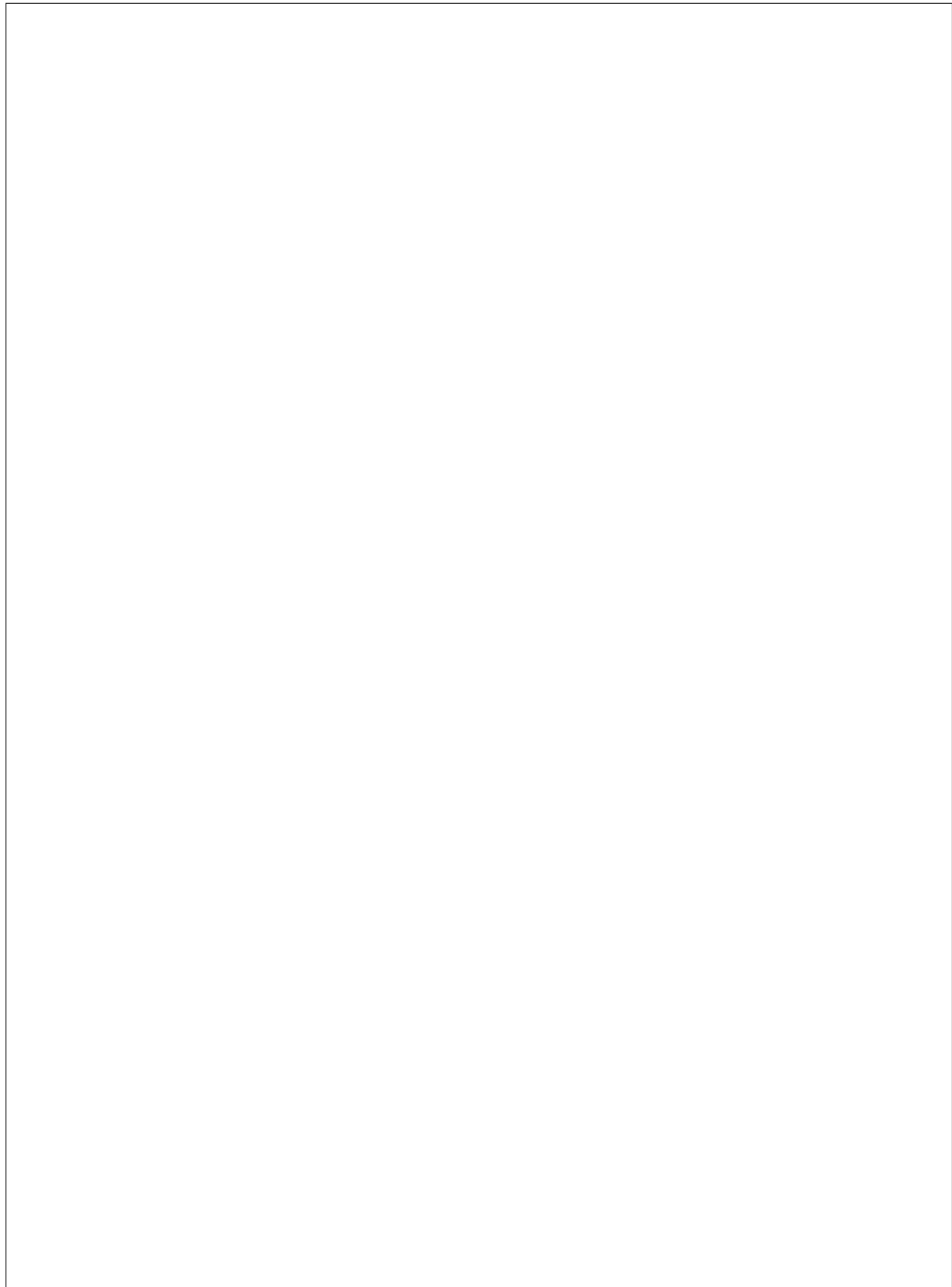
Also, let the continuous random variable Y represent the time (in minutes) that Alex spends at the spot, where Y is uniformly distributed over the interval $[10, 20]$. The appreciation index for the location is defined as the product:

$$Z = X \cdot Y.$$

Answer the following questions:

1. Compute the probability that the appreciation index exceeds 30, i.e., find $\mathbb{P}(Z > 30)$.
2. Derive the cumulative distribution function (CDF) of Z in a piecewise form.
3. Calculate the expected value $\mathbb{E}(Z)$ of the appreciation index.
4. Suppose that Alex visits Wildflower Park twice independently. Let Z_1 and Z_2 be the appreciation indices from each visit. Compute the probability that the appreciation index exceeds 30 on both visits.





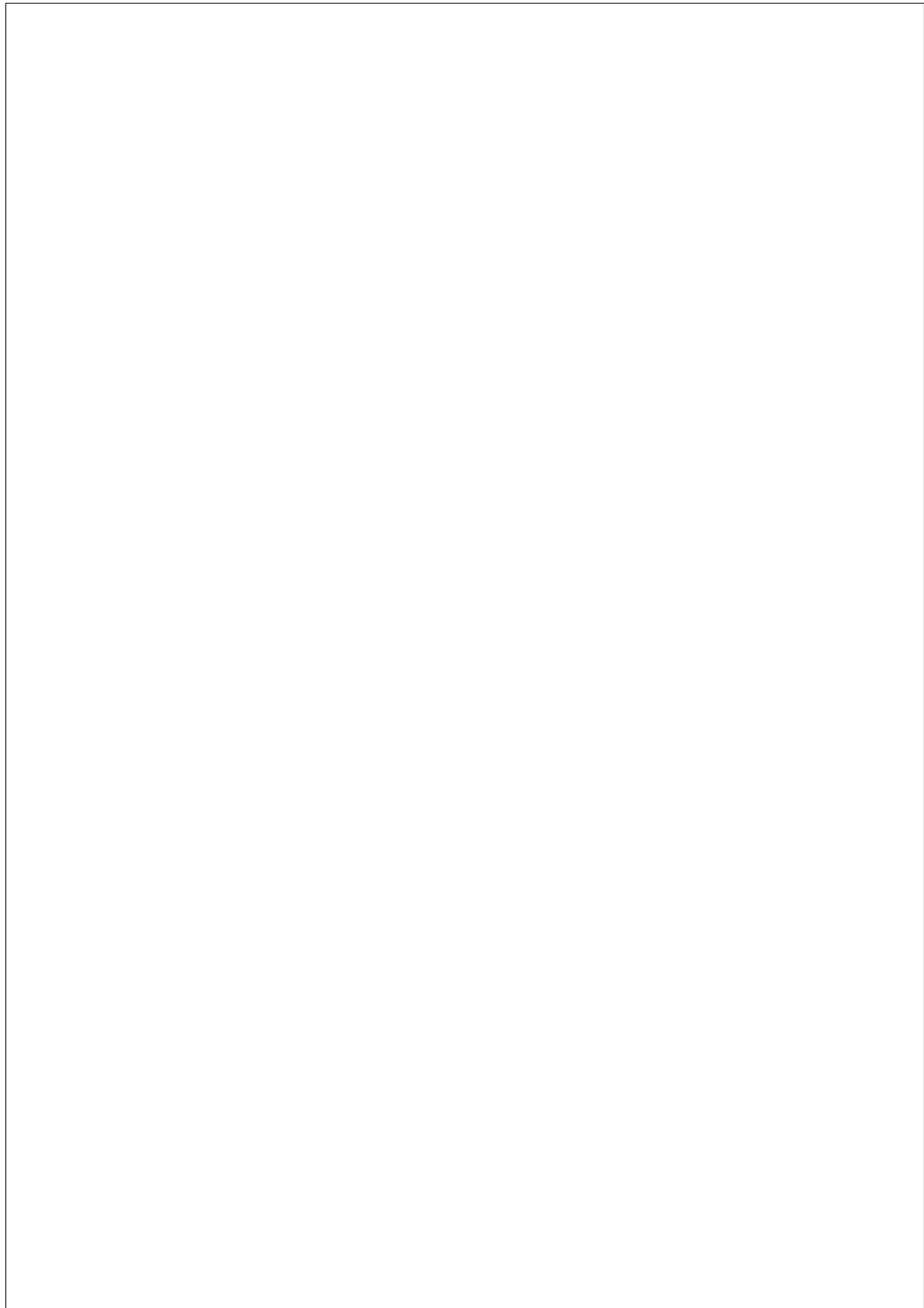
Question 13

The PDF of random variable X is

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 0, \\ \frac{1}{4}, & 0 \leq x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = X^2$ and denote by $F_{X,Y}(x, y)$ the joint CDF of X and Y .

1. Find $f_Y(y)$,
2. Find $F_{X,Y}(-\frac{1}{2}, 4)$.



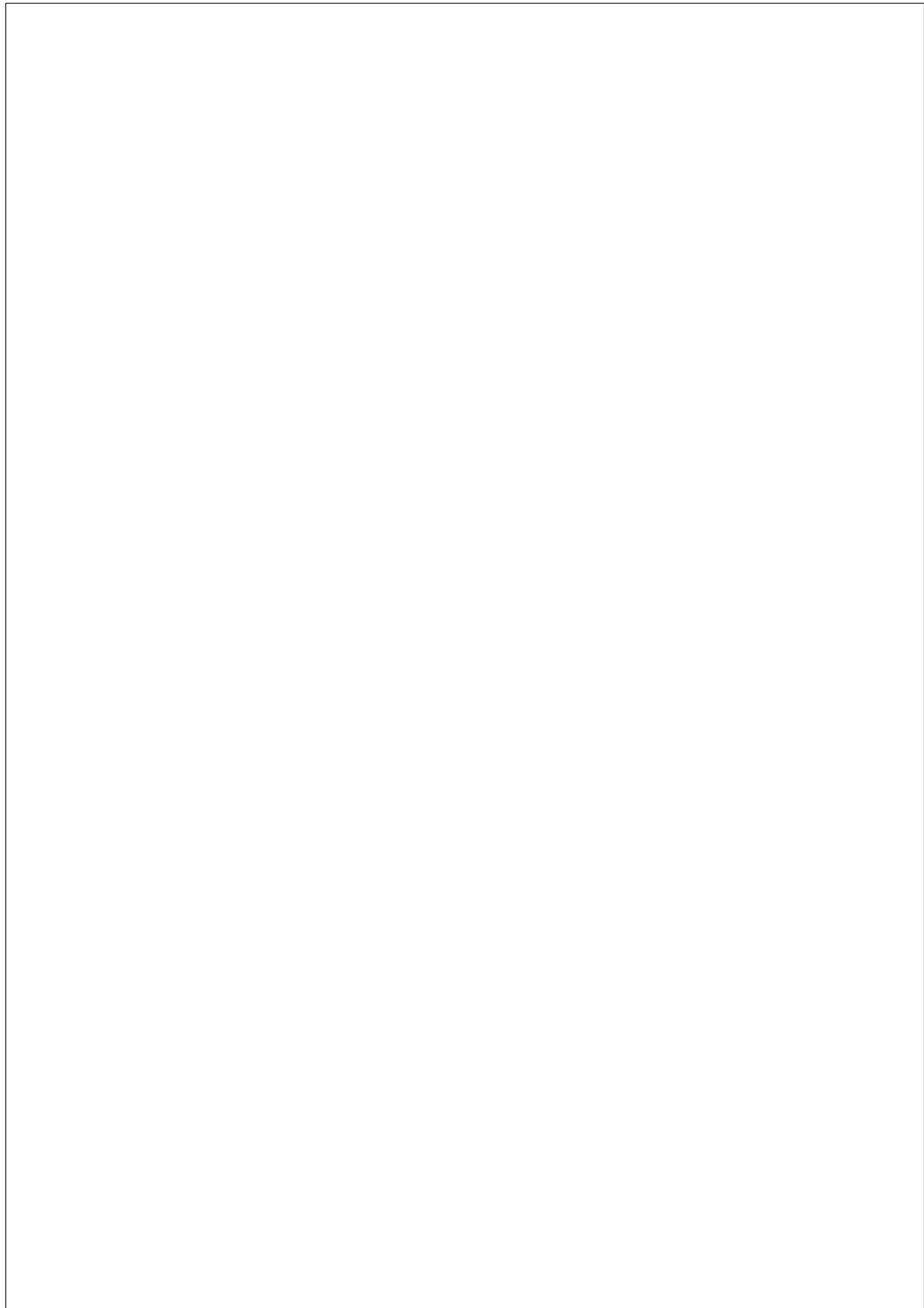
Question 14

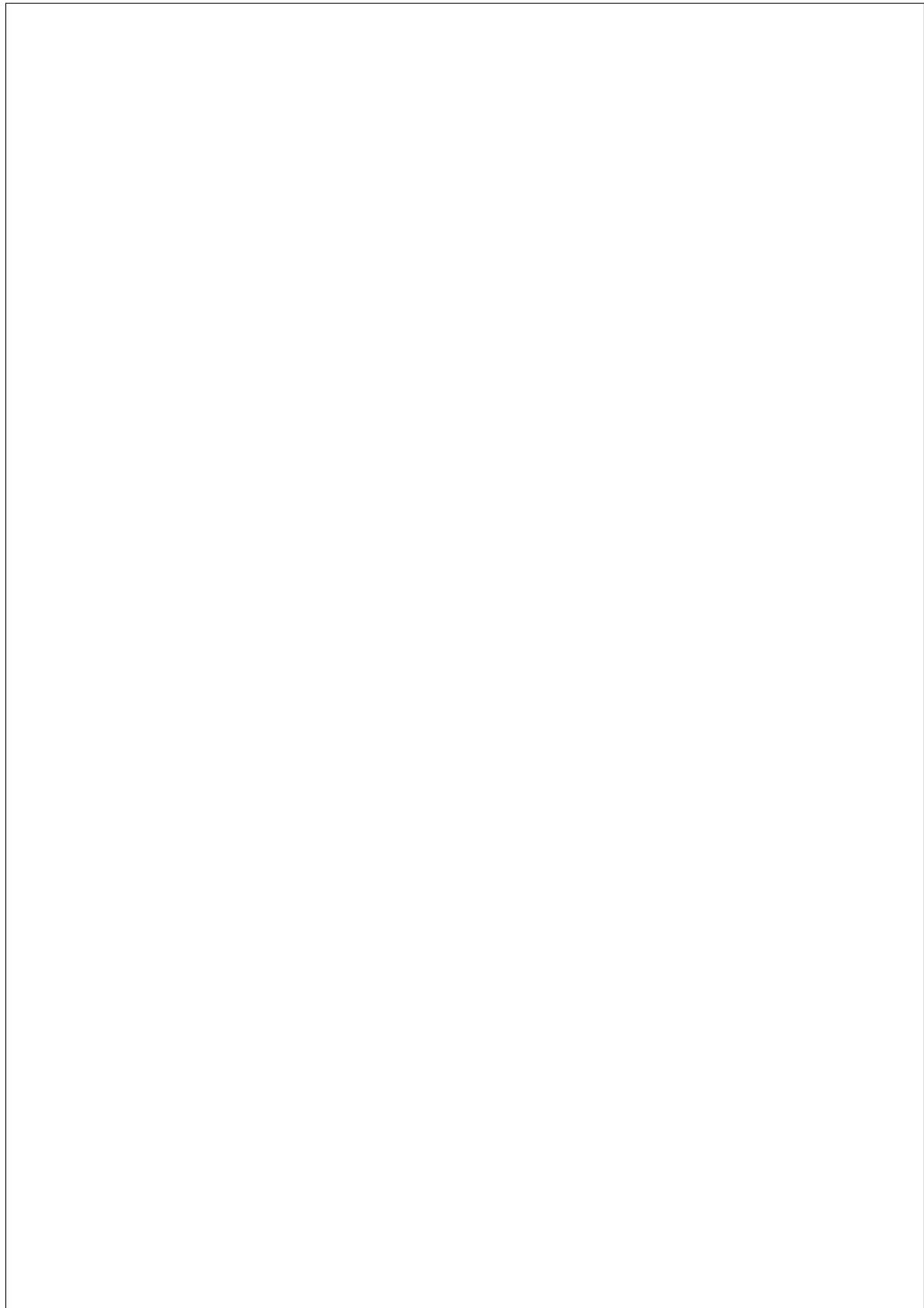
A factory produces two types of electronic components, A and B, with lifetimes denoted as X and Y (in hours), respectively. The joint probability density function (PDF) of their lifetimes is given by:

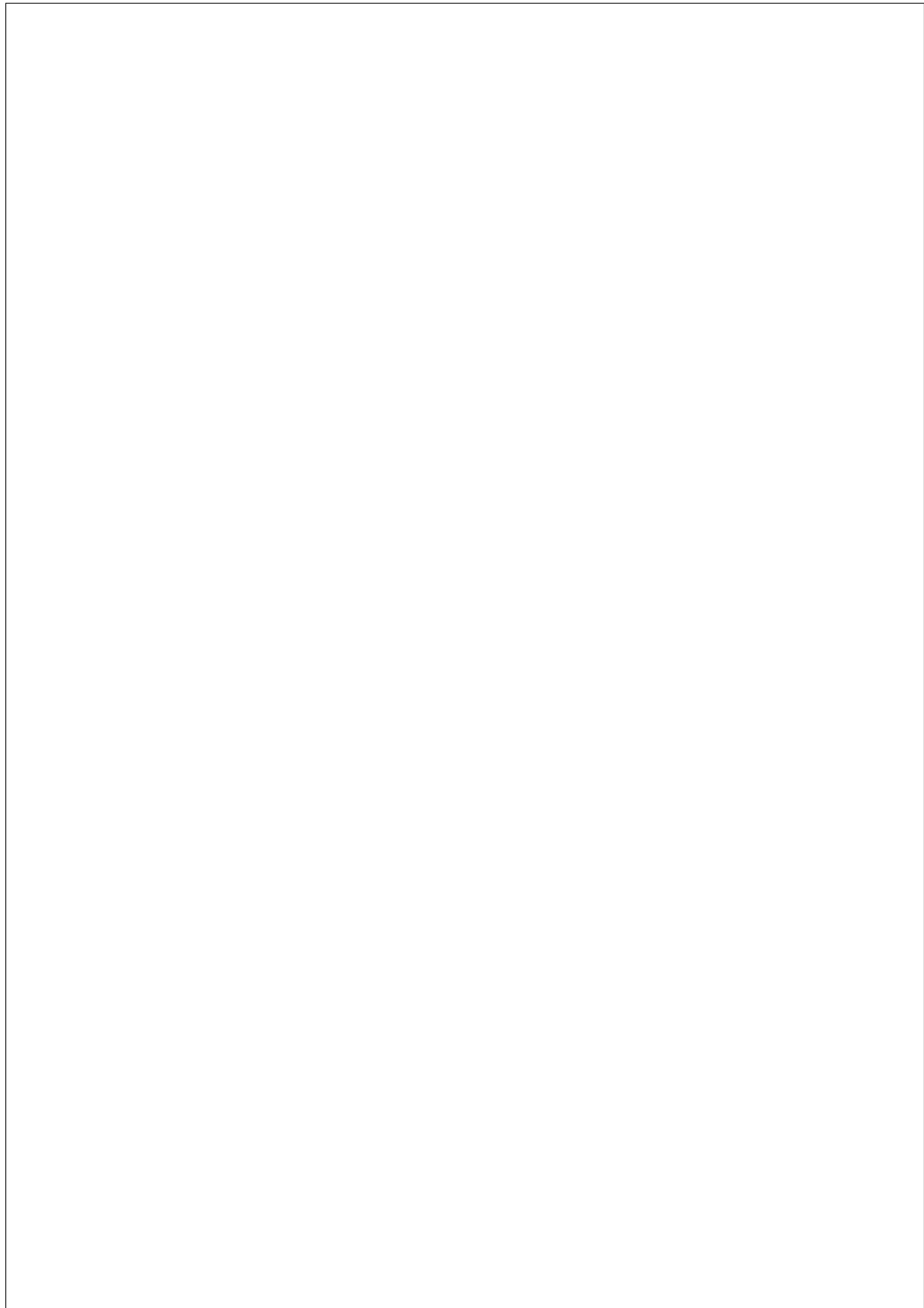
$$f_{X,Y}(x,y) = \begin{cases} ce^{-x-y}, & \text{if } 0 < x < y, \\ ce^{-2x}, & \text{if } 0 < y \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

where c is a constant.

1. Determine the value of the constant c .
2. Find the probability that component A lasts shorter than component B, i.e., $\mathbb{P}(X < Y)$.
3. Given that the total lifetime of a system is the sum of the lifetimes of components A and B, i.e., $Z = X + Y$, compute the probability that the total lifetime Z is less than 1, i.e., $\mathbb{P}(Z < 1)$.
4. Determine whether the lifetimes of the two components are independent.







Question 15

John is a cashier who handles a large number of cash transactions daily. Due to rounding, the amount of each transaction is rounded to the nearest whole dollar. Assume that the rounding error for each transaction is independent and uniformly distributed over the interval $(-0.5, 0.5)$ dollars.

1. If John processes 2 transactions, what is the probability that the absolute value of the total rounding error exceeds 0.5 dollars?
2. If John processes 1500 transactions in a day, what is the probability that the absolute value of the total rounding error exceeds 15 dollars?
3. What is the maximum number of transactions John can process to ensure that the absolute value of the total rounding error is less than 10 dollars with a probability of at least 90%?

