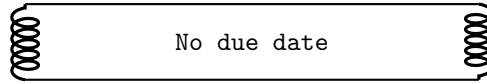


Exercise 2



Instruction: These are for your own practice. There is no need to submit your answer.

Question 1

Let $f_X(x) = \theta x + \frac{1}{2}$ for $-1 \leq x \leq 1$ and zero otherwise, and that θ is a constant.

1. For what value(s) of θ is $f_X(x)$ a valid PDF?
2. For that/those value(s) of θ , find $\mathbb{E}(X)$ and $\text{var}(X)$.
3. For what value(s) of θ is $\text{var}(X)$ maximized?

1. First, let us consider normalization. Note that

$$\int_{-1}^1 \left(\theta x + \frac{1}{2} \right) dx = \left[\frac{1}{2} \theta x^2 + \frac{1}{2} x \right]_{-1}^1 = 1$$

for all values of θ . Another criterion is nonnegativity, i.e., $f_X(x) \geq 0$. This would require $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$.

- 2.

$$\mathbb{E}(X) = \int_{-1}^1 x \left(\theta x + \frac{1}{2} \right) dx = \left[\frac{1}{3} \theta x^3 + \frac{1}{4} x^2 \right]_{-1}^1 = \frac{2}{3} \theta$$

$$\mathbb{E}(X^2) = \int_{-1}^1 x^2 \left(\theta x + \frac{1}{2} \right) dx = \left[\frac{1}{4} \theta x^4 + \frac{1}{6} x^3 \right]_{-1}^1 = \frac{1}{3}$$

$$\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{3} - \frac{4}{9} \theta^2$$

3. The above expression for $\text{var}(X)$ is maximized when $\theta = 0$.

Question 2

A certain object is present with equal likelihood anywhere within a circular region of radius r . Let X be the distance of this object from the center of the region.

1. Find the CDF of X .
2. Find the PDF of X .
3. Find $\mathbb{E}(X)$.
4. Find $\text{var}(X)$.

1. For $0 \leq x \leq r$,

$$F_X(x) = \mathbb{P}(X \leq x) = \frac{\pi x^2}{\pi r^2} = \left(\frac{x}{r}\right)^2$$

Furthermore, $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > r$.

2. Differentiating the above expression,

$$f_X(x) = \begin{cases} \frac{2x}{r^2}, & \text{if } 0 \leq x \leq r \\ 0, & \text{otherwise} \end{cases}$$

3. Using the PDF,

$$\mathbb{E}(X) = \int_0^r x \cdot \frac{2x}{r^2} dx = \frac{2r}{3}$$

4. Similarly,

$$\mathbb{E}(X^2) = \int_0^r x^2 \cdot \frac{2x}{r^2} dx = \frac{r^2}{2}$$

$$\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18}$$

Question 3

The continuous random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2} & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

1. Find the value of c .
2. Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.
3. Let $Y = X^2$. Calculate the conditional expectation and conditional variance of Y given A .

1. Since $\int_1^2 cx^{-2} dx = 1$ due to normalization,

$$c = \frac{1}{\int_1^2 x^{-2} dx} = 2$$

2. We have $\mathbb{P}(A) = \int_{1.5}^2 2x^{-2} dx = \frac{1}{3}$, and therefore

$$f_{X|A}(x|A) = \begin{cases} 6x^{-2} & \text{if } 1.5 < x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

3. We have

$$\mathbb{E}(Y|A) = \mathbb{E}(X^2|A) = \int_{1.5}^2 6x^{-2} x^2 dx = 3$$

$$\mathbb{E}(Y^2|A) = \mathbb{E}(X^4|A) = \int_{1.5}^2 6x^{-2} x^4 dx = \frac{37}{4}$$

$$\text{var}(Y|A) = \frac{37}{4} - 3^2 = \frac{1}{4}$$

Question 4

The joint PDF of random variables X and Y is

$$f(x, y) = \begin{cases} Ce^{-(2x+4y)} & x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

1. Compute C .
2. Compute the joint CDF $F_{X,Y}(x, y)$.
3. Compute $\mathbb{P}(Y \leq X)$.
4. Compute the conditional PDF $f_{Y|X}(y|x)$.

1. The maximum CDF should be 1, i.e., $F_{X,Y}(\infty, \infty) = 1$.

$$1 = \iint_{x>0, y>0} f(x, y) dx dy = \iint_{x>0, y>0} Ce^{-(2x+4y)} dx dy = C \int_0^{\infty} e^{-2x} dx \int_0^{\infty} e^{-4y} dy = \frac{C}{8},$$

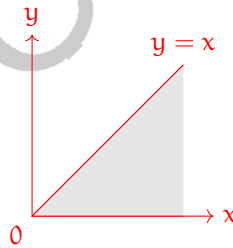
So $C = 8$.

2. As in the last question,

$$F_{X,Y}(x, y) = \begin{cases} \int_{-\infty}^x \int_{-\infty}^y 8e^{-(2x+4y)} dx dy = (1 - e^{-2x})(1 - e^{-4y}) & x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that you need to specify the expression $(1 - e^{-2x})(1 - e^{-4y})$ is only applicable for the quadrant $x > 0, y > 0$.

3. The region of integration is below the line $y = x$ in the following diagram.



$$\mathbb{P}(Y \leq X) = \int_0^{\infty} \int_0^x 8e^{-(2x+4y)} dx dy = \frac{2}{3}.$$

4. The marginal distribution of X is

$$f_X(x) = \int_0^{\infty} e^{-2x-4y} dy = 2e^{-2x}, \quad x > 0,$$

Then the conditional PDF is

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \begin{cases} \frac{8e^{-(2x+4y)}}{2e^{-2x}} = 4e^{-4y} & y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Question 5

Suppose X is a continuous uniform random variable between 0 and 1, i.e., $X \sim U(0, 1)$. Find the PDF of $Y = e^X$.

Since $0 \leq x \leq 1$, we know that $1 \leq y \leq e$. Following our standard procedure,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(e^X \leq y) = \mathbb{P}(X \leq \ln(y)) = \ln(y)$$

Differentiating the CDF with respect to y , then,

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 \leq y \leq e \\ 0 & \text{otherwise.} \end{cases}$$

Question 6

We have three independent exponential random variables X , Y , and W , all with parameters λ .

1. What is the PDF of $Z = X + Y$?
2. What is the PDF of $Z = X + Y + W$?

1. We have

$$f_X(x) = f_Y(y) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Making use of the convolution formula, for $z > 0$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-x) f_Y(x) dx \\ &= \int_0^z \lambda e^{-\lambda(z-x)} \cdot \lambda e^{-\lambda x} dx \\ &= \lambda^2 \int_0^z e^{-\lambda z} dx \\ &= \lambda^2 z e^{-\lambda z} \end{aligned}$$

For $z < 0$, $f_Z(z) = 0$. Combining the two cases,

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z} & z \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. Again we make use of the convolution formula, for $z > 0$,

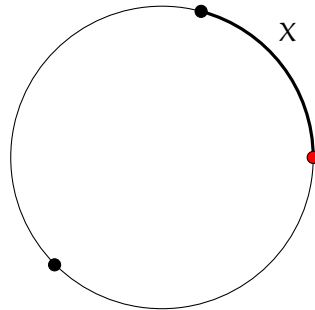
$$\begin{aligned} f_Z(z) &= \int_0^z \lambda e^{-\lambda(z-x)} \cdot \lambda^2 x e^{-\lambda x} dx \\ &= \lambda^3 e^{-\lambda z} \int_0^z x dx \\ &= \frac{1}{2} \lambda^3 z^2 e^{-\lambda z} \end{aligned}$$

For $z < 0$, $f_Z(z) = 0$.

We can show that the sum of n independent exponential distributions follow $f_Z(z) = \frac{(\lambda z)^{n-1}}{(n-1)!} \lambda e^{-\lambda z}$.

Question 7

Assume three points, one of which is red, are uniformly and independently distributed over the circumference of a circular track, which has a total length of 1. Equivalently, you can interpret the circle as a unit interval with its two ends tied together, and all distances of points are measured in terms of the length on the track. The Figure below gives a simple illustration.



1. Let X be the distance of the red point to its neighbor in the counter-clockwise direction. Find the PDF of X .
2. Let Y be the minimum distance of the red point to its nearest neighbor (either in the clockwise or counter-clockwise direction, whichever is smaller). Find the PDF of Y .
3. Let Z be the minimum distance between any two of the points. Find the PDF of Z .

Hint: You can represent the event $\{Z > z\}$ in terms of two random variables that are uniformly distributed on a unit square. To find $\mathbb{P}(Z > z)$, determine the area that corresponds to $\{Z > z\}$ graphically.

1. Let us denote by A and B the random variables representing the counter-clockwise distances of the two black points from the red point. We will map the circle to a unit interval starting at the position of the red dot. Note that A and B are uniformly distributed in this interval independently of each other.

To compute the pdf of X we first compute its CDF,

$$\begin{aligned} F_X(x) &= \mathbb{P}(X \leq x) = 1 - \mathbb{P}(X > x) = 1 - \mathbb{P}(A > x, B > x) \\ &= 1 - \mathbb{P}(A > x) \mathbb{P}(B > x) \\ &= 1 - (1 - x)^2 \end{aligned}$$

Differentiating, $f_X(x) = \begin{cases} 2(1 - x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

2. Since we are “wrapping around the circle”, the value of Y can only be between 0 and $1/2$, with a uniform distribution. Therefore, for point A to have a distance greater than y from the red dot, the probability is now $(1 - 2y)$.

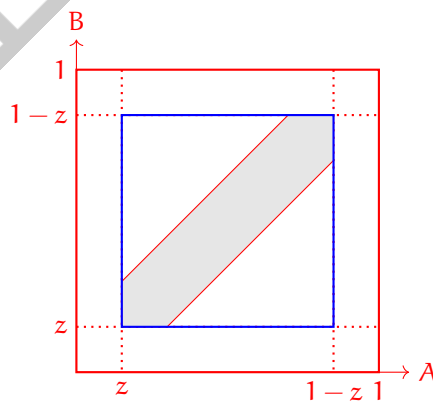
$$F_Y(y) = \mathbb{P}(Y \leq y) = 1 - \mathbb{P}(Y > y) = 1 - \mathbb{P}(A > y) \mathbb{P}(B > y) = 1 - (1 - 2y)^2$$

Differentiating, $f_Y(y) = \begin{cases} 4(1 - 2y) & 0 \leq y \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$

3. The value of Z can only be between 0 and $1/3$. For the event $\{Z > z\}$, we require:

- $z < A < 1 - z$
- $z < B < 1 - z$
- $|A - B| > z$

Let us try to visualize the region graphically. In the Figure below, for a certain value of z , the blue square satisfies the first two conditions. We then need to subtract the gray area in order to satisfy the third condition. A little geometry argument finds that the remaining two triangles have sides $(1 - 3z)$. Hence, $\mathbb{P}(Z > z) = (1 - 3z)^2$, and $\mathbb{P}(Z \leq z) = 1 - (1 - 3z)^2$.



Differentiating, $f_Z(z) = \begin{cases} 6(1 - 3z) & 0 \leq z \leq \frac{1}{3} \\ 0 & \text{otherwise.} \end{cases}$

Question 8

The PDF of the Rayleigh random variable is given by

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find:

1. the CDF;
2. the mean;
3. the median;
4. the mode.

1. The CDF:

$$F_X(x) = \int_0^x \frac{r}{\sigma^2} e^{-r^2/(2\sigma^2)} dr$$

Let $y = \frac{r^2}{2\sigma^2}$. Then, $dy = \frac{r dr}{\sigma^2}$. Moreover, when $r = 0$, we have $y = 0$; when $r = x$, we have $y = \frac{x^2}{2\sigma^2}$. Therefore,

$$F_X(x) = \int_0^{x^2/2\sigma^2} e^{-y} dy = [-e^{-y}]_0^{x^2/2\sigma^2} = 1 - e^{-x^2/(2\sigma^2)}$$

2. The mean:

$$\begin{aligned} \mathbb{E}(X) &= \int_0^\infty (1 - F_X(x)) dx = \int_0^\infty e^{-x^2/(2\sigma^2)} dx \\ &= \sigma\sqrt{2\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)} dx \\ &= \sigma\sqrt{2\pi} \cdot \frac{1}{2} \\ &= \sigma\sqrt{\frac{\pi}{2}} \end{aligned}$$

because $\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)} dx$ is the area under the positive side of a zero-mean Gaussian.

3. The median:

$$\begin{aligned} 1 - e^{-x^2/(2\sigma^2)} &= \frac{1}{2} \\ -x^2/(2\sigma^2) &= \ln \frac{1}{2} \\ x^2 &= 2\sigma^2 \ln 2 \\ x &= \sigma\sqrt{2 \ln 2} \end{aligned}$$

4. The mode: We find the maximum of $f_X(x)$ by taking the derivative with respect to x and setting it to zero:

$$\frac{d}{dx} \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} = \frac{1}{\sigma^2} e^{-x^2/(2\sigma^2)} - \frac{x}{\sigma^2} \cdot \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} = 0$$

This gives $x^2 = \sigma^2$, and therefore, $x = \sigma$.

Question 9

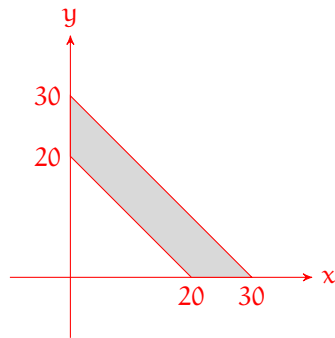
A dairy farm produces two different types of milk. Let X = the amount (in liter) of type A on hand and Y = the amount of type B on hand. Suppose the joint PDF of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} kxy & x \geq 0, \quad y \geq 0, \quad 20 \leq x+y \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

where k is a certain constant to be determined.

1. Draw the region of positive density, and determine the value of k .
2. Are X and Y independent variables? Derive the marginal PDF of each variable.
3. Compute $\mathbb{P}(X+Y) \leq 25$.
4. What is the expected total amount of milk on hand? (correct to one decimal place)

1. The figure should include two lines $x + y = 30$ and $x + y = 20$, in the region of positive coordinates ($x, y \geq 0$).



$$1 = \iint_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = \int_0^{20} \int_{20-x}^{30-x} kxy \, dy \, dx + \int_{20}^{30} \int_0^{30-x} kxy \, dy \, dx = \frac{81250}{3}k$$

$$k = \frac{3}{81250}$$

2. The marginal distribution of X is

$$f_X(x) = \begin{cases} \int_{20-x}^{30-x} kxy \, dy = k(250x - 10x^2) & 0 \leq x \leq 20 \\ \int_0^{30-x} kxy \, dy = k(450x - 30x^2 + \frac{1}{2}x^3) & 20 \leq x \leq 30 \end{cases}$$

and by symmetry $f_Y(y)$ is obtained by substituting y for x in $f_X(x)$. We note, for example, that $f_X(25) > 0$ and $f_Y(25) > 0$, but $f_{X,Y}(25, 25) = 0$; therefore, $f_X(x) f_Y(y) \neq f_{X,Y}(x, y)$, so X and Y are not independent.

- 3.

$$\mathbb{P}(X + Y \leq 25) = \int_0^{20} \int_{20-x}^{25-x} kxy \, dy \, dx + \int_{20}^{25} \int_0^{25-x} kxy \, dy \, dx = \frac{3}{81250} \cdot \frac{230625}{24} = 0.355$$

- 4.

$$\begin{aligned} \mathbb{E}(X + Y) &= \mathbb{E}(X) + \mathbb{E}(Y) = 2 \left(\int_0^{20} xk(250x - 10x^2) \, dx + \int_{20}^{30} xk(450x - 30x^2 + \frac{1}{2}x^3) \, dx \right) \\ &= 2k \cdot 351666.67 \approx 26.0 \end{aligned}$$

Question 10

A dart lands uniformly random on a circular target of radius r . Let x and y be the Cartesian coordinates. Thus, we can view x to be a realization of a random variable X , and y a realization of another random variable Y .

1. Are X and Y independent?
2. Are X and Y uncorrelated?

1. The joint PDF is $f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise.} \end{cases}$

The marginal PDF $f_Y(y)$ is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \frac{1}{\pi r^2} \int_{x^2 + y^2 \leq r^2} dx \\ &= \frac{1}{\pi r^2} \int_{-\sqrt{r^2 - y^2}}^{\sqrt{r^2 - y^2}} dx \\ &= \frac{2}{\pi r^2} \sqrt{r^2 - y^2} \quad \text{if } |y| \leq r \end{aligned}$$

The marginal PDF $f_X(x)$ is similar. Since $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$, the two random variables are not independent.

2. We need to compute the covariance $\text{cov}(X, Y)$. From the above, by symmetry, we see that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$. We now need to find $\mathbb{E}(XY)$. This is equal to

$$\begin{aligned} \mathbb{E}(XY) &= \iint_{-\infty}^{\infty} xy f_{X,Y}(x,y) dx dy \\ &= \frac{1}{\pi r^2} \iint_{x^2 + y^2 \leq r^2} xy dx dy \end{aligned}$$

We can convert it to polar coordinates and compute the integration, but it is also sufficient to note that by symmetry, the integration will come out to be zero.

Thus, $\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0$, and therefore X and Y are uncorrelated.

Question 11

A dart lands uniformly random within a rhombus-shaped target bounded by the following four equations, where x and y are the Cartesian coordinates:

$$x + 2y = 2 \quad \text{for } x \geq 0, y \geq 0$$

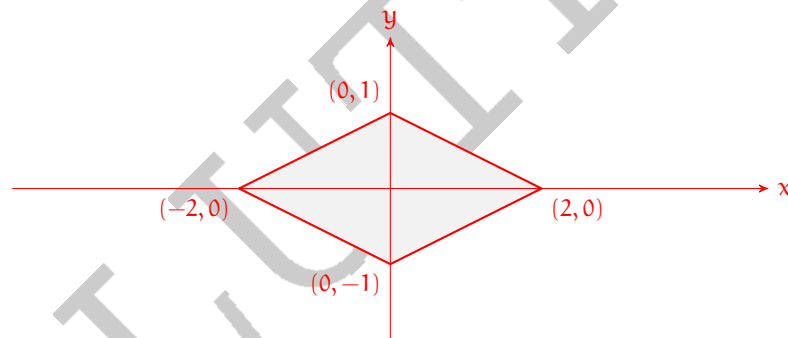
$$x - 2y = -2 \quad \text{for } x < 0, y \geq 0$$

$$x + 2y = -2 \quad \text{for } x < 0, y < 0$$

$$x - 2y = 2 \quad \text{for } x \geq 0, y < 0$$

1. Sketch the shape of the target. Mark clearly the coordinates of the corners.
2. Find the marginal PDF $f_Y(y)$.
3. Find the conditional PDF $f_{X|Y}(x|y)$.

1. The target is as follows:



2. We can do that geometrically. The total area of the rhombus is 4. Therefore, $f_{X,Y}(x,y) = \frac{1}{4}$ within the target, which we can represent by the equation

$$|x| + 2|y| \leq 2.$$

For $0 \leq y \leq 1$, we have $x = 4(1 - y)$. For $-1 \leq y < 0$, we have $x = 4(1 + y)$. Combining, we can write $x = 4 - 4|y|$. But this is not yet the marginal PDF, as we have to normalize to ensure that the total area under the PDF is one. The marginal PDF is therefore

$$f_Y(y) = \begin{cases} 1 - |y| & -1 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

If you prefer to use integration rather than geometric argument, we can proceed as follows. Consider first $y \geq 0$.

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \int_{2y-2}^{2-2y} \frac{1}{4} dx \\ &= \frac{1}{4} [(2 - 2y) - (2y - 2)] \\ &= 1 - y \end{aligned}$$

The case for $y < 0$ is similar.

3. The conditional PDF, provided $|y| < 1$, is

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{\frac{1}{4}}{1 - |y|} \\ &= \frac{1}{4(1 - |y|)} \quad \text{if } |x| \leq 2(1 - |y|) \end{aligned}$$

Given a fixed y , this is uniform!

Question 12

Alex, an avid hiker, visits Wildflower Park, renowned for its breathtaking scenery and abundant wildflowers. In one of the open areas, Alex not only counts the number of wildflowers he observes but also records the time (in minutes) he spends at the spot. Let the discrete random variable X denote the number of wildflowers observed by Alex on a single visit, with possible values 0, 1, 2, and 3 having the probability mass function:

$$\mathbb{P}(X = 0) = 0.1, \quad \mathbb{P}(X = 1) = 0.4, \quad \mathbb{P}(X = 2) = 0.3, \quad \mathbb{P}(X = 3) = 0.2.$$

Also, let the continuous random variable Y represent the time (in minutes) that Alex spends at the spot, where Y is uniformly distributed over the interval $[10, 20]$. The appreciation index for the location is defined as the product:

$$Z = X \cdot Y.$$

Answer the following questions:

1. Compute the probability that the appreciation index exceeds 30, i.e., find $\mathbb{P}(Z > 30)$.
2. Derive the cumulative distribution function (CDF) of Z in a piecewise form.
3. Calculate the expected value $\mathbb{E}(Z)$ of the appreciation index.
4. Suppose that Alex visits Wildflower Park twice independently. Let Z_1 and Z_2 be the appreciation indices from each visit. Compute the probability that the appreciation index exceeds 30 on both visits.

1. We condition on X .

- When $X = 0$: $Z = 0$, so $\mathbb{P}(Z > 30) = 0$.
- When $X = 1$: $Z = Y$ with $Y \in [10, 20]$. Since $Y \leq 20 < 30$, so $\mathbb{P}(Z > 30) = 0$.
- When $X = 2$: $Z = 2Y$. The inequality $2Y > 30$ implies $Y > 15$. Since $Y \sim U(10, 20)$,

$$\mathbb{P}(Y > 15) = \frac{20 - 15}{20 - 10} = \frac{5}{10} = 0.5.$$

- When $X = 3$: $Z = 3Y$. The inequality $3Y > 30$ implies $Y > 10$. As $Y \geq 10$, therefore $\mathbb{P}(Y > 10) = 1$.

By the law of total probability,

$$\mathbb{P}(Z > 30) = \mathbb{P}(X = 2) \cdot 0.5 + \mathbb{P}(X = 3) \cdot 1 = 0.3 \cdot 0.5 + 0.2 \cdot 1 = 0.15 + 0.2 = 0.35.$$

2. CDF of Z :

We derive the cumulative distribution function (CDF) of Z by conditioning on X . Note that:

- For $X = 0$: $Z = 0$ deterministically, so

$$\mathbb{P}(Z \leq z | X = 0) = \begin{cases} 0, & z < 0, \\ 1, & z \geq 0. \end{cases}$$

- For $X = 1$: $Z = Y$ with $Y \sim \mathcal{U}(10, 20)$, hence

$$\mathbb{P}(Z \leq z | X = 1) = \begin{cases} 0, & z < 10, \\ \frac{z - 10}{10}, & 10 \leq z < 20, \\ 1, & z \geq 20. \end{cases}$$

- For $X = 2$: $Z = 2Y$ with $Z \in [20, 40]$, so

$$\mathbb{P}(Z \leq z | X = 2) = \begin{cases} 0, & z < 20, \\ \frac{\frac{z}{2} - 10}{10}, & 20 \leq z < 40, \\ 1, & z \geq 40. \end{cases}$$

- For $X = 3$: $Z = 3Y$ with $Z \in [30, 60]$, hence

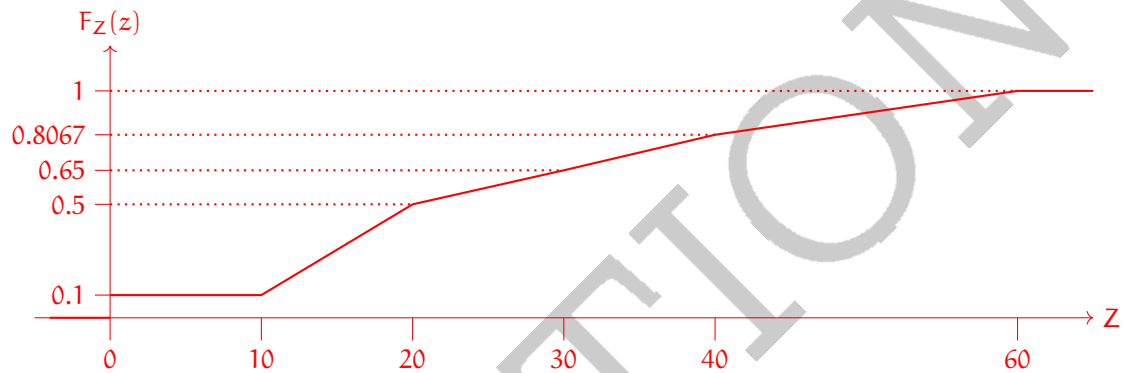
$$\mathbb{P}(Z \leq z | X = 3) = \begin{cases} 0, & z < 30, \\ \frac{\frac{z}{3} - 10}{10}, & 30 \leq z < 60, \\ 1, & z \geq 60. \end{cases}$$

Using the law of total probability, for $z \geq 0$,

$$F_Z(z) = \mathbb{P}(Z \leq z) = \sum_x \mathbb{P}(X = x) \mathbb{P}(Z \leq z | X = x).$$

Hence, the CDF of Z is given by:

$$F_Z(z) = \begin{cases} 0, & z < 0, \\ 0.1, & 0 \leq z < 10, \\ 0.1 + 0.4 \frac{z-10}{10}, & 10 \leq z < 20, \\ 0.1 + 0.4 + 0.3 \frac{\frac{z}{2}-10}{10}, & 20 \leq z < 30, \\ 0.1 + 0.4 + 0.3 \frac{\frac{z}{2}-10}{10} + 0.2 \frac{\frac{z}{3}-10}{10}, & 30 \leq z < 40, \\ 0.1 + 0.4 + 0.3 + 0.2 \frac{\frac{z}{3}-10}{10}, & 40 \leq z < 60, \\ 1, & z \geq 60. \end{cases}$$



3. Expected value of Z :

Since X and Y are independent,

$$\mathbb{E}(Z) = \mathbb{E}(X \cdot Y) = \mathbb{E}(X) \mathbb{E}(Y).$$

We have

$$\mathbb{E}(X) = 0 \cdot 0.1 + 1 \cdot 0.4 + 2 \cdot 0.3 + 3 \cdot 0.2 = 1.6,$$

and for $Y \sim U(10, 20)$,

$$\mathbb{E}(Y) = \frac{10 + 20}{2} = 15.$$

Thus,

$$\mathbb{E}(Z) = 1.6 \times 15 = 24.$$

4. Probability that $Z > 30$ on both visits:

Let Z_1 and Z_2 be the appreciation indices from two independent visits. Then, by independence,

$$\mathbb{P}(Z_1 > 30 \text{ and } Z_2 > 30) = \mathbb{P}(Z > 30)^2.$$

From part (1), since $\mathbb{P}(Z > 30) = 0.35$, it follows that

$$\mathbb{P}(Z_1 > 30 \text{ and } Z_2 > 30) = (0.35)^2 = 0.1225.$$

Question 13

The PDF of random variable X is

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 0, \\ \frac{1}{4}, & 0 \leq x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = X^2$ and denote by $F_{X,Y}(x, y)$ the joint CDF of X and Y .

1. Find $f_Y(y)$,
2. Find $F_{X,Y}(-\frac{1}{2}, 4)$.

1. Since $Y = X^2$, the CDF of Y can be written as

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 \leq y).$$

Note that:

- For $y < 0$, we have $F_Y(y) = \mathbb{P}(Y \leq y) = 0$ since the full range of Y is $[0, +\infty)$.
- For $0 \leq y < 1$, we have

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X^2 \leq y) \\ &= \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \mathbb{P}(-\sqrt{y} \leq X < 0) + \mathbb{P}(0 \leq X \leq \sqrt{y}) \\ &= \frac{1}{2}\sqrt{y} + \frac{1}{4}\sqrt{y} = \frac{3}{4}\sqrt{y}. \end{aligned}$$

- For $1 \leq y < 4$, we have

$$\begin{aligned} F_Y(y) &= \mathbb{P}(-\sqrt{y} \leq X < 0) + \mathbb{P}(0 \leq X \leq \sqrt{y}) \\ &= \mathbb{P}(-1 \leq X < 0) + \mathbb{P}(0 \leq X \leq \sqrt{y}) \\ &= \frac{1}{2} + \frac{1}{4}\sqrt{y}. \end{aligned}$$

- For $y \geq 4$, we have

$$\begin{aligned} F_Y(y) &= \mathbb{P}(-\sqrt{y} \leq X < 0) + \mathbb{P}(0 \leq X \leq \sqrt{y}) \\ &= \mathbb{P}(-1 \leq X < 0) + \mathbb{P}(0 \leq X < 2) \\ &= 1. \end{aligned}$$

In conclusion, we have

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{3}{4}\sqrt{y}, & 0 \leq y < 1, \\ \frac{1}{2} + \frac{1}{4}\sqrt{y}, & 1 \leq y < 4, \\ 1, & y \geq 4. \end{cases} \implies f_Y(y) = \begin{cases} \frac{3}{8\sqrt{y}}, & 0 \leq y < 1, \\ \frac{1}{8\sqrt{y}}, & 1 \leq y < 4, \\ 0, & \text{otherwise.} \end{cases}$$

2. We have

$$\begin{aligned} F_{X,Y}\left(-\frac{1}{2}, 4\right) &= \mathbb{P}\left(X \leq -\frac{1}{2}, Y \leq 4\right) \\ &= \mathbb{P}\left(X \leq -\frac{1}{2}, X^2 \leq 4\right) \\ &= \mathbb{P}\left(X \leq -\frac{1}{2}, -2 \leq X \leq 2\right) \\ &= \mathbb{P}\left(-2 \leq X \leq -\frac{1}{2}\right) \\ &= \mathbb{P}(-2 \leq X \leq -1) + \mathbb{P}\left(-1 < X \leq -\frac{1}{2}\right) \\ &= 0 + \frac{1}{4} = \frac{1}{4}. \end{aligned}$$

Question 14

A factory produces two types of electronic components, A and B, with lifetimes denoted as X and Y (in hours), respectively. The joint probability density function (PDF) of their lifetimes is given by:

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x-y}, & \text{if } 0 < x < y, \\ ce^{-2x}, & \text{if } 0 < y \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

where c is a constant.

1. Determine the value of the constant c .
2. Find the probability that component A lasts shorter than component B, i.e., $\mathbb{P}(X < Y)$.
3. Given that the total lifetime of a system is the sum of the lifetimes of components A and B, i.e., $Z = X + Y$, compute the probability that the total lifetime Z is less than 1, i.e., $\mathbb{P}(Z < 1)$.
4. Determine whether the lifetimes of the two components are independent.

The solution will start on the next page.

1. Determine the value of the constant c :

The joint PDF must satisfy the normalization condition:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \, dy = 1.$$

We split the integral into two regions:

- Region 1: $0 < x < y$

$$\begin{aligned} \int_0^{\infty} \int_x^{\infty} c e^{-x-y} \, dy \, dx &= c \int_0^{\infty} e^{-x} \left(\int_x^{\infty} e^{-y} \, dy \right) dx \\ &= c \int_0^{\infty} e^{-x} \cdot e^{-x} \, dx = c \int_0^{\infty} e^{-2x} \, dx \\ &= c \left[-\frac{1}{2} e^{-2x} \right]_0^{\infty} = \frac{c}{2}. \end{aligned}$$

- Region 2: $0 < y \leq x$

$$\begin{aligned} \int_0^{\infty} \int_0^x c e^{-2x} \, dy \, dx &= c \int_0^{\infty} e^{-2x} \cdot x \, dx \\ &= c \left[-\frac{1}{4} (2x+1) e^{-2x} \right]_0^{\infty} = \frac{c}{4}. \end{aligned}$$

Adding both regions:

$$\frac{c}{2} + \frac{c}{4} = \frac{3c}{4} = 1 \implies c = \frac{4}{3}.$$

Answer: $c = \frac{4}{3}$.

2. Find the probability $\mathbb{P}(X < Y)$:

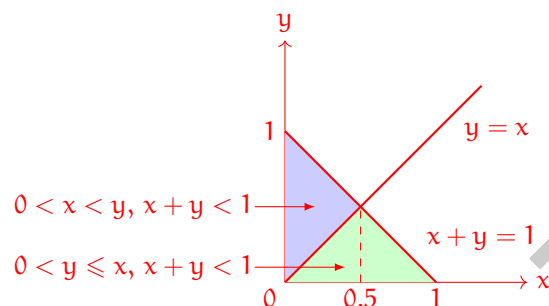
This probability is the integral over the region $0 < x < y$:

$$\mathbb{P}(X < Y) = \int_0^{\infty} \int_x^{\infty} \frac{4}{3} e^{-x-y} \, dy \, dx = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3}.$$

Answer: $\mathbb{P}(X < Y) = \frac{2}{3}$.

3. Compute $\mathbb{P}(Z < 1)$

We need to find $\mathbb{P}(X + Y < 1)$. The region of integration is $0 < x < 1$ and $0 < y < 1 - x$.



- For $0 < x < y$ and $x + y < 1$:

$$\begin{aligned}
 \int_0^{0.5} \int_x^{1-x} \frac{4}{3} e^{-x-y} dy dx &= \frac{4}{3} \int_0^{0.5} e^{-x} \left(\int_x^{1-x} e^{-y} dy \right) dx \\
 &= \frac{4}{3} \int_0^{0.5} e^{-x} (e^{-x} - e^{-(1-x)}) dx \\
 &= \frac{4}{3} \int_0^{0.5} (e^{-2x} - e^{-1}) dx \\
 &= \frac{4}{3} \left[-\frac{1}{2} e^{-2x} - x e^{-1} \right]_0^{0.5} \\
 &= \frac{2}{3} (1 - 2e^{-1}).
 \end{aligned}$$

- For $0 < y \leq x$ and $x + y < 1$:

$$\begin{aligned}
 &\int_0^{0.5} \int_0^x \frac{4}{3} e^{-2x} dy dx + \int_{0.5}^1 \int_0^{1-x} \frac{4}{3} e^{-2x} dy dx \\
 &= \frac{4}{3} \int_0^{0.5} x e^{-2x} dx + \frac{4}{3} \int_{0.5}^1 (1-x) e^{-2x} dx \\
 &= \frac{4}{3} \left[-\frac{1}{4} (2x+1) e^{-2x} \right]_0^{0.5} + \frac{4}{3} \left[\frac{1}{4} (2x-1) e^{-2x} \right]_{0.5}^1 \\
 &= \frac{1}{3} (1 - 2e^{-1}) + \frac{1}{3} e^{-2}.
 \end{aligned}$$

Therefore,

$$\mathbb{P}(Z < 1) = 1 - 2e^{-1} + \frac{1}{3}e^{-2}.$$

4. Determine whether X and Y are independent:

To check for independence, we need to verify if $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$.

Derive the marginal PDFs:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy = \int_x^{\infty} \frac{4}{3} e^{-x-y} \, dy + \int_0^x \frac{4}{3} e^{-2x} \, dy \\ &= \frac{4}{3} [-e^{-x-y}]_x^{\infty} + \frac{4}{3} [ye^{-2x}]_0^x \\ &= \frac{4}{3} e^{-2x}(x+1), \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx = \int_0^y \frac{4}{3} e^{-x-y} \, dx + \int_y^{\infty} \frac{4}{3} e^{-2x} \, dx \\ &= \frac{4}{3} [-e^{-x-y}]_0^y + \frac{4}{3} \left[-\frac{1}{2} e^{-2x} \right]_y^{\infty} \\ &= \frac{4}{3} e^{-y} (1 - e^{-y}) + \frac{2}{3} e^{-2y} \\ &= \frac{4}{3} e^{-y} - \frac{2}{3} e^{-2y}. \end{aligned}$$

It is clear that:

$$f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y).$$

Answer: X and Y are not independent.

Question 15

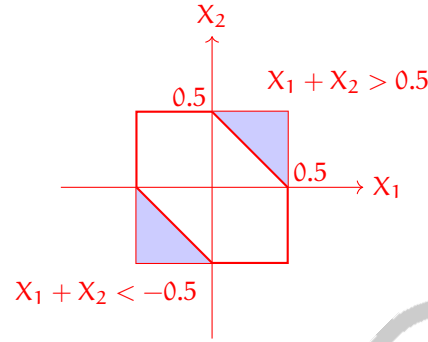
John is a cashier who handles a large number of cash transactions daily. Due to rounding, the amount of each transaction is rounded to the nearest whole dollar. Assume that the rounding error for each transaction is independent and uniformly distributed over the interval $(-0.5, 0.5)$ dollars.

1. If John processes 2 transactions, what is the probability that the absolute value of the total rounding error exceeds 0.5 dollars?
2. If John processes 1500 transactions in a day, what is the probability that the absolute value of the total rounding error exceeds 15 dollars?
3. What is the maximum number of transactions John can process to ensure that the absolute value of the total rounding error is less than 10 dollars with a probability of at least 90%?

The solution will start on the next page.

Let the rounding error of the k -th transaction be X_k ($k = 1, 2, \dots, n$).

1. Let $S = X_1 + X_2$ be the total rounding error of 2 transactions. Since X_1 and X_2 are independent of each other, so their joint PDF is equal to the product of their individual PDFs. We can visualize the joint PDF and then tackle this problem geometrically.



As shown above, the joint PDF of X_1 and X_2 is constant and equal to 1 within the region where $X_1 \in (-0.5, 0.5)$ and $X_2 \in (-0.5, 0.5)$. The requirement $|X_1 + X_2| > 0.5$ corresponds to the blue shaded area in the graph. Therefore, we can calculate as follows:

$$\begin{aligned}\mathbb{P}(|S| > 0.5) &= \mathbb{P}(|X_1 + X_2| > 0.5) \\ &= \mathbb{P}(X_1 + X_2 > 0.5) + \mathbb{P}(X_1 + X_2 < -0.5) \\ &= 2 \times 0.5 \times 0.5 \times 0.5 = 0.25.\end{aligned}$$

2. It is known that $X_k \sim \mathcal{U}(-0.5, 0.5)$, so we have $\mathbb{E}(X_k) = 0$ and $\text{var}(X_k) = \frac{1}{12}$.

Let $X = \sum_{k=1}^{1500} X_k$. By the Central Limit Theorem, when n is sufficiently large, we have the approximation:

$$\mathbb{P}\left(\frac{\sum_{k=1}^{1500} X_k - 1500 \times 0}{\sqrt{1500} \sqrt{1/12}} \leq x\right) \approx \Phi(x).$$

Thus,

$$\begin{aligned}\mathbb{P}(|X| > 15) &= 1 - \mathbb{P}(|X| \leq 15) = 1 - \mathbb{P}(-15 \leq X \leq 15) \\ &= 1 - \mathbb{P}\left(\frac{-15 - 0}{\sqrt{1500} \sqrt{1/12}} \leq \frac{X - 0}{\sqrt{1500} \sqrt{1/12}} \leq \frac{15 - 0}{\sqrt{1500} \sqrt{1/12}}\right) \\ &\approx 1 - \left[\Phi\left(\frac{15}{\sqrt{1500} \sqrt{1/12}}\right) - \Phi\left(\frac{-15}{\sqrt{1500} \sqrt{1/12}}\right)\right] \\ &= 1 - \left[2\Phi\left(\frac{15}{\sqrt{1500/12}}\right) - 1\right] = 1 - [2\Phi(1.342) - 1] \\ &= 2[1 - 0.9099] = 0.1802.\end{aligned}$$

That is, the probability that the absolute value of the total rounding error exceeds 15 dollars is approximately 0.1802.

3. Let the maximum number of transactions be m , such that the total error $Y = \sum_{k=1}^m X_k$ satisfies the condition:

$$\mathbb{P}(|Y| < 10) \geq 0.90.$$

By the Central Limit Theorem, when m is sufficiently large, we have the approximation:

$$\mathbb{P}\left(\frac{Y - m \times 0}{\sqrt{m}\sqrt{1/12}} \leq x\right) \approx \Phi(x).$$

Thus,

$$\begin{aligned} \mathbb{P}(|Y| < 10) &= \mathbb{P}(-10 < Y < 10) \\ &= \mathbb{P}\left(\frac{-10}{\sqrt{m}\sqrt{1/12}} < \frac{Y}{\sqrt{m}\sqrt{1/12}} < \frac{10}{\sqrt{m}\sqrt{1/12}}\right) \\ &\approx \Phi\left(\frac{10}{\sqrt{m/12}}\right) - \Phi\left(\frac{-10}{\sqrt{m/12}}\right) \\ &= 2\Phi\left(\frac{10}{\sqrt{m/12}}\right) - 1. \end{aligned}$$

Therefore, m must satisfy:

$$2\Phi\left(\frac{10}{\sqrt{m/12}}\right) - 1 \geq 0.90,$$

which is equivalent to:

$$\Phi\left(\frac{10}{\sqrt{m/12}}\right) \geq 0.95 = \Phi(1.645),$$

and thus:

$$\frac{10}{\sqrt{m/12}} \geq 1.645.$$

Solving for m , we get:

$$m \leq 443.45.$$

Since m must be a positive integer, the value of m is 443.