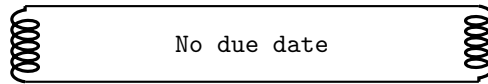


Exercise 1



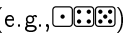
Instruction: These are for your own practice. There is no need to submit your answer.

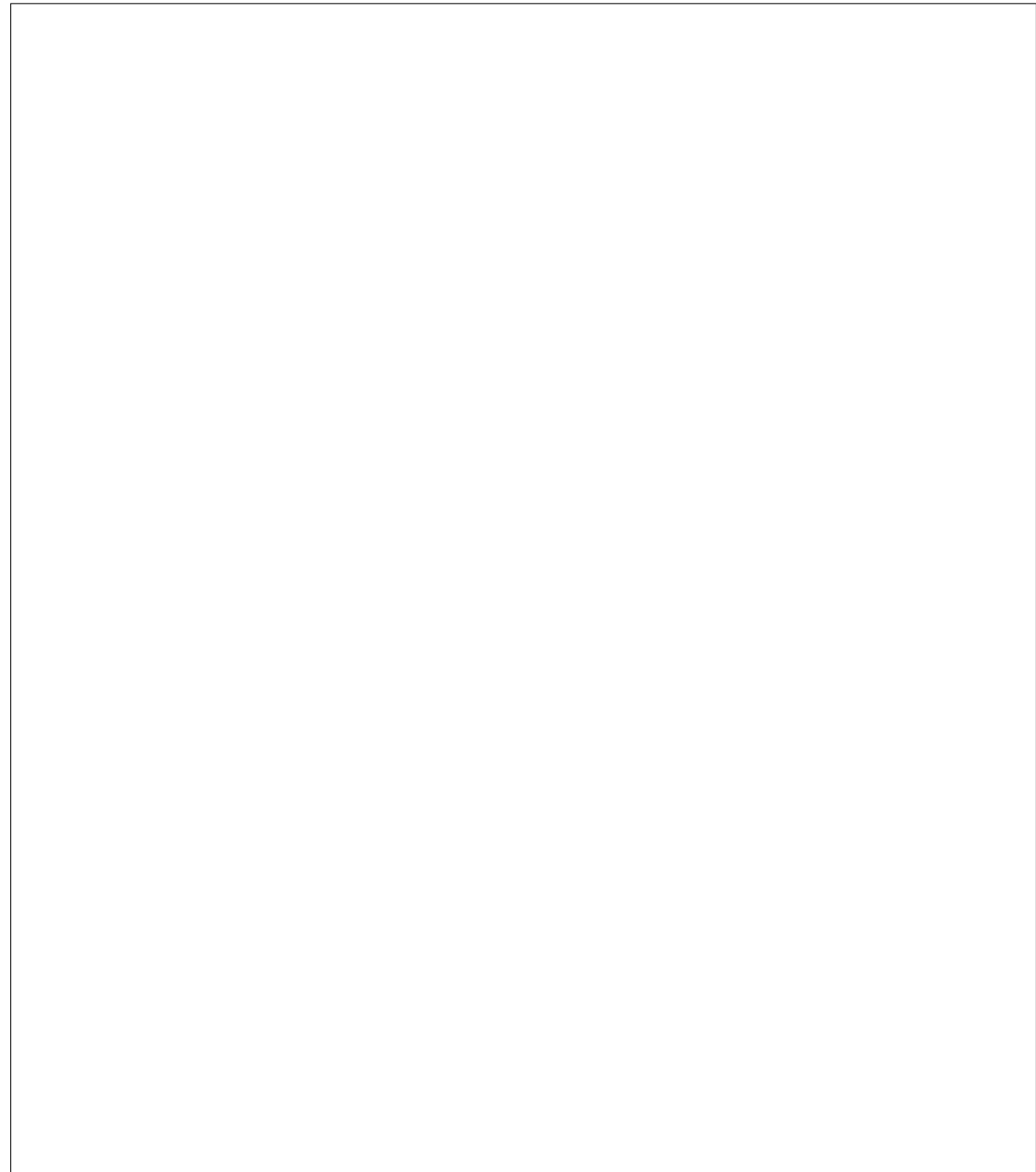
Question 1 Assessing risk

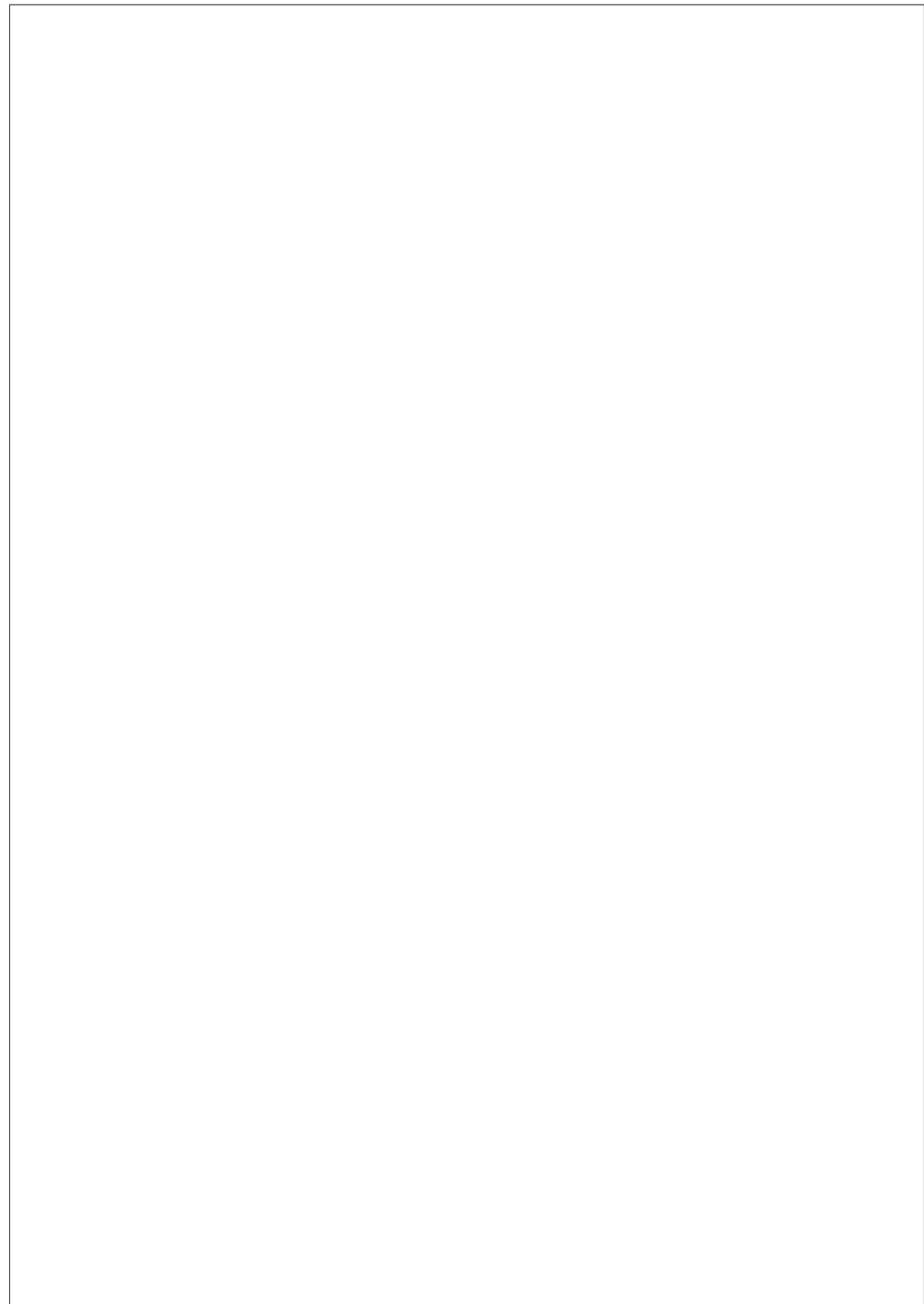
In the boardgame called Risk[®], at a given time, a player is the attacker (“A”) and another player is the defender (“D”) of a territory. The outcome of the attack is determined by the following set of rules:

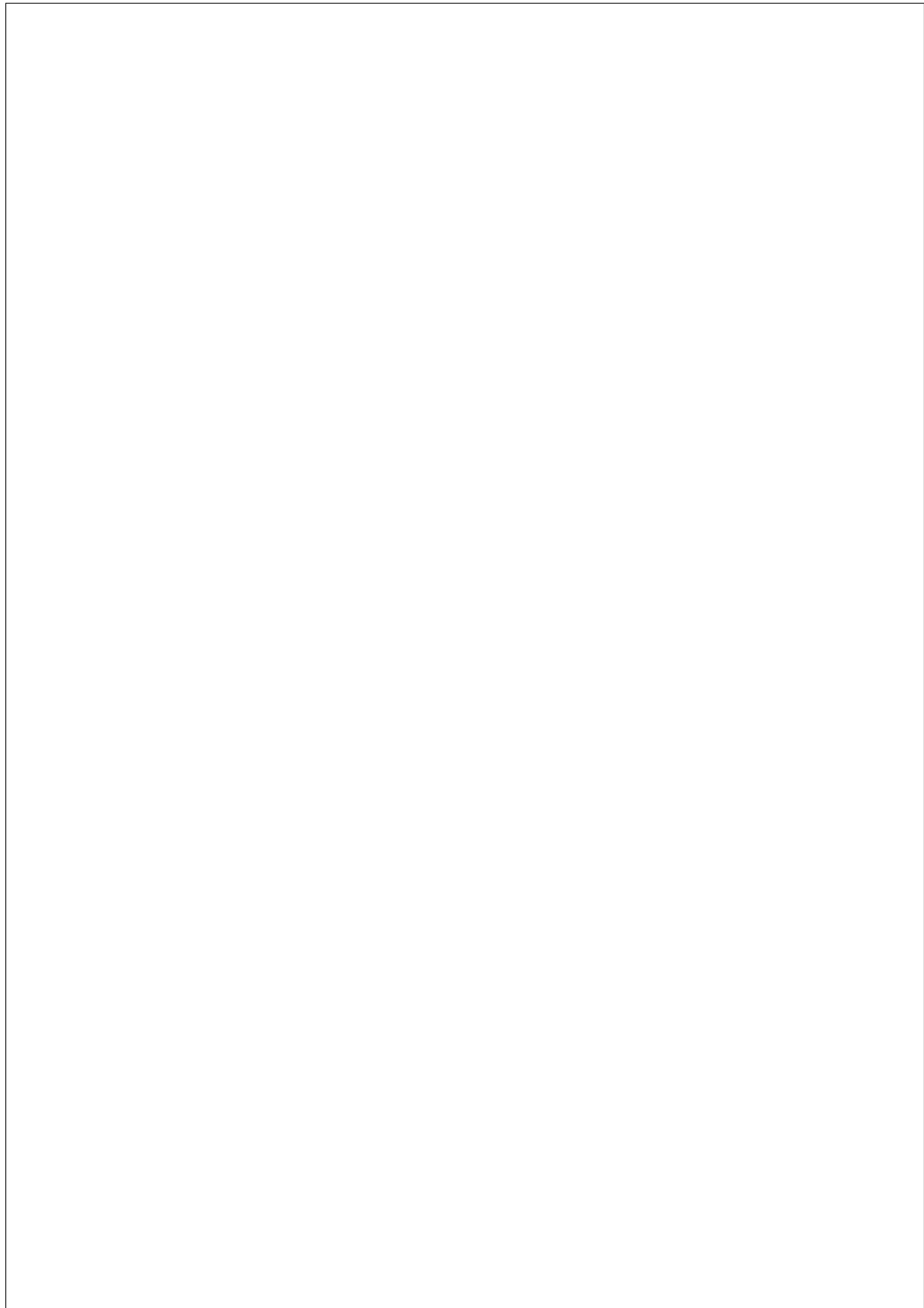
- (I) “A” rolls between one and three dice.
- (II) “D” rolls between one and two dice.
- (III) The highest rolled die from “A” is compared to the highest rolled die from “D”:
 - (a) “A” loses (one combat) if its value is *smaller than or equal to* that from “D”;
 - (b) “A” wins that combat otherwise.
- (IV) The compared dice are discarded, and if both players still have dice left, repeat Step (III) above.

We would like to analyze the odds of winning and losing in different situations.

1. Suppose A rolls one die (e.g., ) and D rolls one die (e.g., ). What is the probability of A succeeding in its attack?
2. Suppose A rolls two dice (e.g., ) and D rolls one die (e.g., ). What is the probability of A succeeding in its attack?
3. Suppose A rolls three dice (e.g., ) and D rolls one die (e.g., ). What is the probability of A succeeding in its attack?
4. Suppose A rolls one die (e.g., ) and D rolls two dice (e.g., ). What is the probability of A succeeding in its attack?
5. (Optional, bonus) Suppose A rolls two dice (e.g., ) and D also rolls two dice (e.g., ).
 - (a) What is the probability of A winning both combats?
 - (b) What is the probability of A winning one combat and losing the other?







Question 2

Assume that we have a well-shuffled deck of cards, and we then get five cards among the 52. What is the probability that it is a “full house”? This is defined as having a three-of-a-kind (i.e., the same number or rank, but different suits) and a pair, e.g., you get the nine of hearts, nine of spades, nine of diamonds, king of hearts, king of clubs.

Question 3

You are a contestant in a game show, in which there is a prize hidden behind one of three doors with equal likelihood. You are allowed to pick one door. The host, who knows where the prize is, then opens one of the remaining two doors and reveals that there is no prize behind that door. At this point, you have two options:

1. Stick to your original choice.
2. Switch to the other door that has not yet been opened.

Would option 1 give you better odds, or option 2, or they are the same? Explain.

Question 4

We have defined “conditional probability” as

$$\mathbb{P}(A|B) \triangleq \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad (1)$$

(assuming $\mathbb{P}(B) \neq 0$) but how do we know that this expression indeed gives a “probability”? Remember from the course notes that there are three axioms that all probability functions must satisfy. Show that $\mathbb{P}(A|B)$ indeed complies with these axioms.

Question 5

Two students, A and B, toss a biased coin successively, where the probability of head is $\frac{1}{3}$. A tosses first, then B, and then the sequence repeats again starting with A. The person who tosses the first head wins. Find the probability of winning for each.

Question 6

In a game, the players need to control a game character to defeat as many monsters as possible. The players will only encounter one monster each round. A player's chance of defeating a monster in each round is p . If the player fails to defeat the monster for r rounds in total, the game ends.

1. Denote by X the number of rounds in which this player successfully defeats the monster before the game ends. What is the PMF of X ?
2. What is the expected value of X ?

Question 7

Let X be a discrete random variable that takes on the integer values from 0 to 9 with equal probability. Also, the notation $r = a \pmod{b}$ means r is the remainder when a is divided by b .

1. Find the PMF of Y , where $Y = X \pmod{3}$.
2. Find the PMF of Z , where $Z = 5 \pmod{(X + 1)}$.

Question 8

The random variable X has the following PMF

$$p_X(x) = \begin{cases} |x|/a & \text{if } x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}.$$

1. Find a and $\mathbb{E}(X)$.
2. What is the PMF of the random variable $Z = (X - \mathbb{E}(X))^2$?
3. Using the result above, find the variance of X .
4. Find the variance of X using the formula $\text{var}(X) = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$?

Question 9

Suppose that X and Y are independent geometric random variables, both with parameter p . Show that

$$\mathbb{P}(X = i | X + Y = n) = \frac{1}{n-1} \quad i = 1, \dots, n-1.$$

Note that this is a discrete uniform distribution, which has the same probability as long as i is within the prescribed range.

Question 10

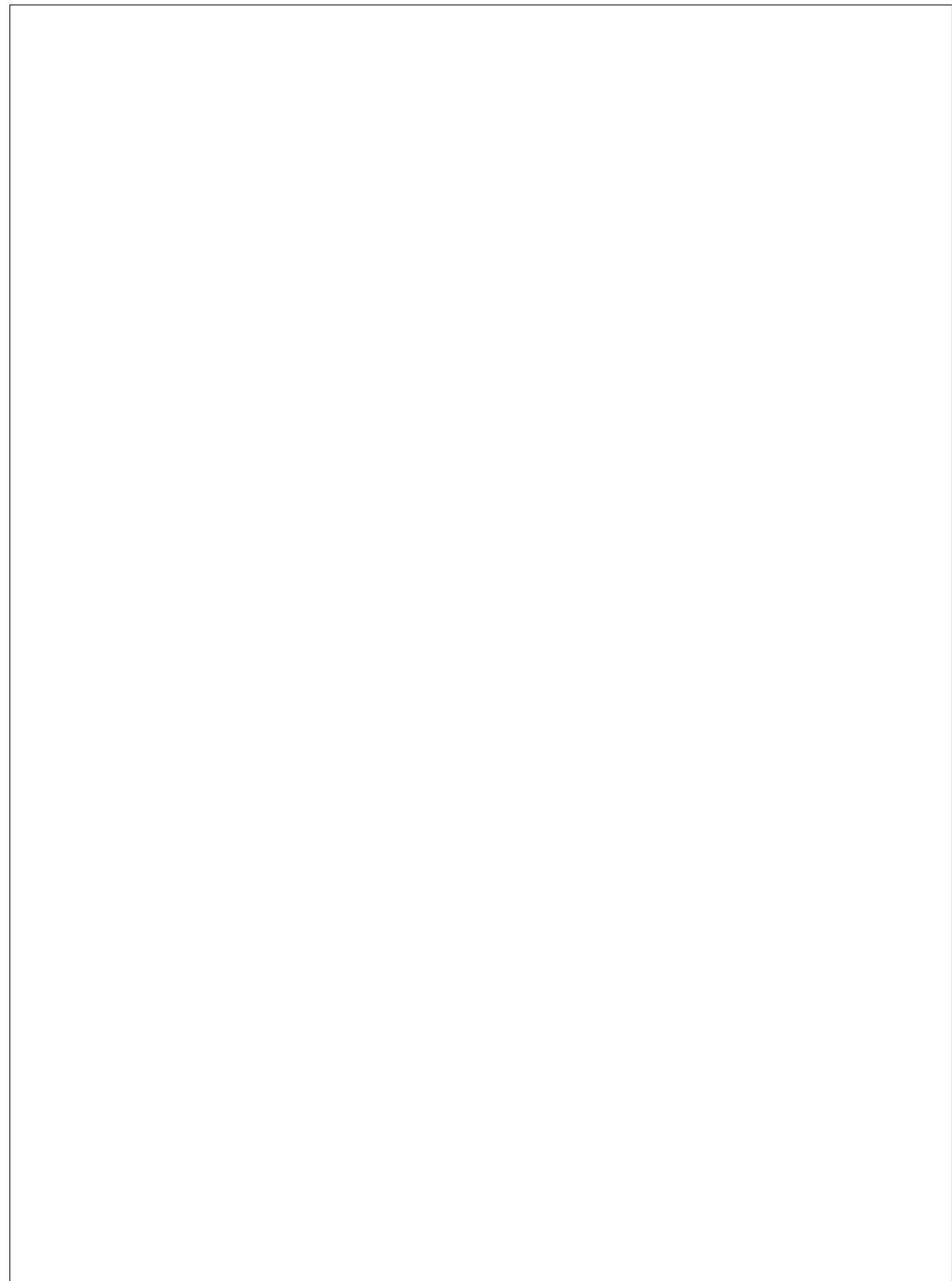
The number of crossways a taxi passes through in a day follows a Poisson distribution, i.e., $Y \sim \text{Poisson}(\lambda)$. Assume that the traffic lights at each crossway work independently and the probability of encountering a red light at each crossway is $p > 0$.

1. If a taxi passes through k crossways in a day, what is the distribution of the number of red lights it encounters?
2. What is the distribution of the number of red lights a taxi encounters in one day?

Question 11

A person goes to a store to buy cups. After collecting 20 cups with a box, he wants to verify their quality. Rather than checking each cup individually, he plans to randomly examine 4 cups.

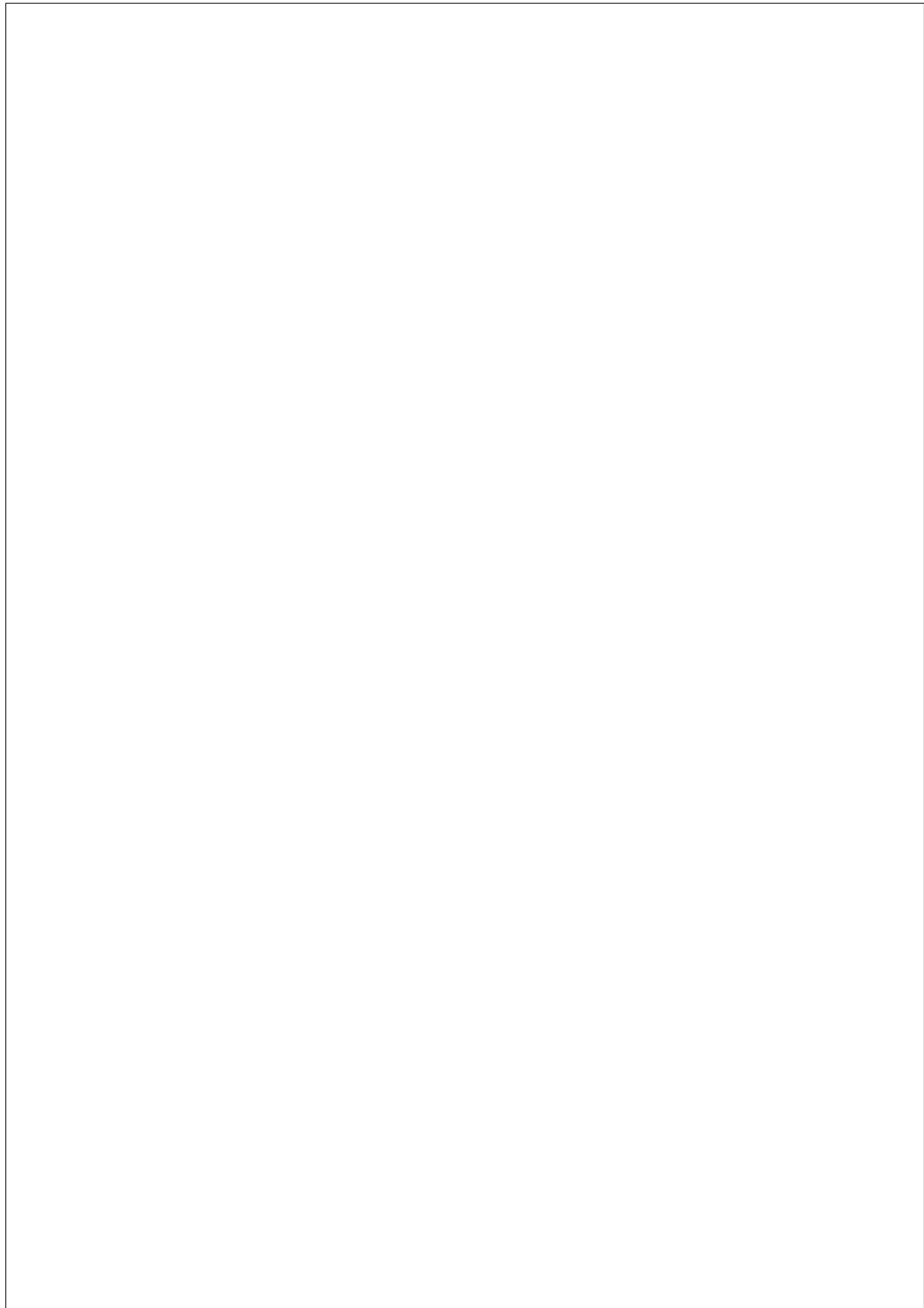
1. Find the probability of there being no quality problem with the 4 selected cups under different assumptions as follows:
 - (a) Assume that the probability of each cup in the box being with quality problem is 0.1.
 - (b) Assume that 2 cups in this box are with quality problems.
 - (c) Assume that the probabilities of 0, 1, and 2 cups in this box having quality problems are 0.8, 0.1, and 0.1, respectively.
2. Given that the person has checked 4 cups and found no problem, find the probability that the other 16 cups are all in good quality under different assumptions as follows:
 - (a) Assume that the probability of each cup in the box being with quality problem is 0.1.
 - (b) Assume that the probabilities of 0, 1, and 2 cups in this box having quality problems are 0.8, 0.1, and 0.1, respectively.

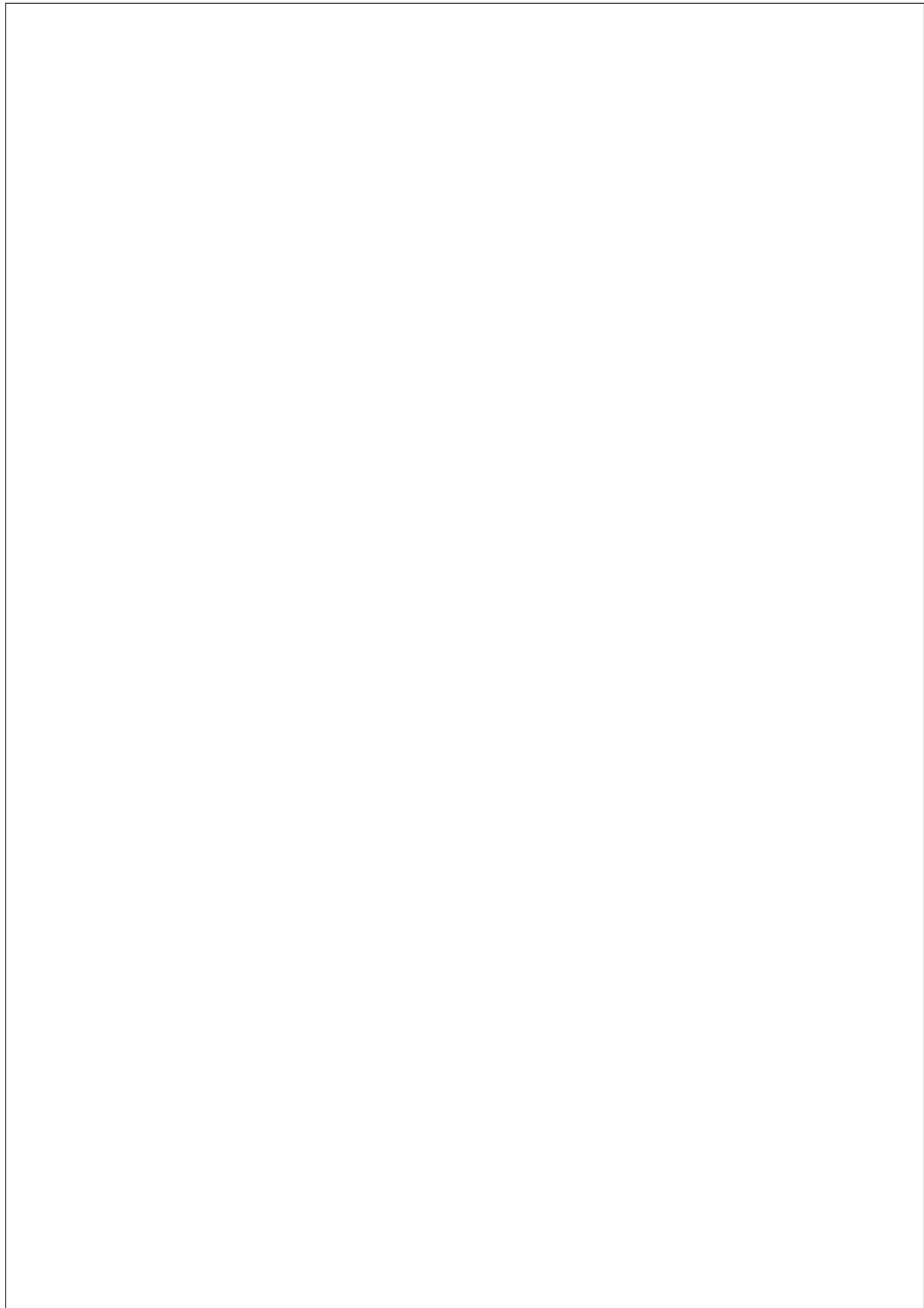


Question 12

There is a room with 4 windows of the same size, only one of which is open. A pigeon flies into the room through the open window and can only fly away through the open window. Now the pigeon flies around the room, trying to escape. Assume that the pigeon has no memory and flies randomly towards each window.

1. Let X denote the number of attempts the pigeon makes to escape the room.
 - (a) Successfully choosing the open window or not is a Bernoulli trial. What is the probability of the Bernoulli trial? What kind of distribution does X follow?
 - (b) Let k denote the number of needed trials. Express the probability mass function (PMF) of X .
2. The owner claims that his pigeon has memory and will not try the same window more than once. Let Y denote the number of attempts this smart pigeon makes to escape the room.
 - (a) Find all possible values of Y and explain the situations for each one.
 - (b) Let i denote the number of needed trials. Express the PMF of Y .
3.
 - (a) Find the probability that the number of attempts X is less than Y (i.e., $\mathbb{P}(X < Y)$).
 - (b) Find the probability that Y is less than X (i.e., $\mathbb{P}(Y < X)$).

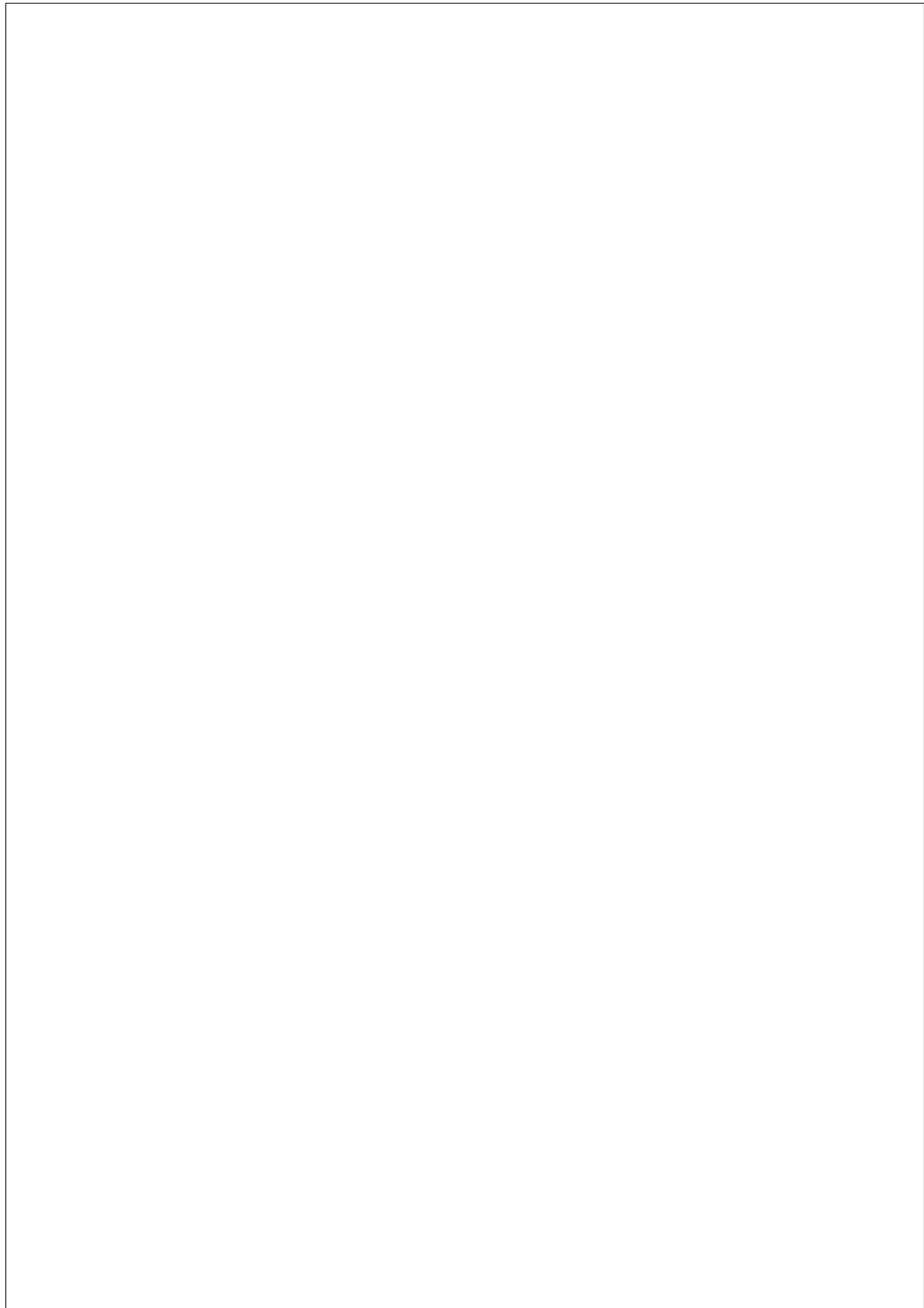




Question 13

Every week, Rachel is tasked with organizing 4 community events. Let X represent the number of events she successfully organizes each week, which follows a Poisson distribution with $\lambda = 6$.

1. What is the probability that Rachel successfully organizes enough events for a week?
2. Let Y denote the participants' boredom index for the week. The relationship between Y and X is given by: $Y = 1/(X + 1)$. Calculate the expectation of Y .
3. Unfortunately, each week there is a 40% chance that a special guest speaker is visiting, and during those weeks Rachel is only able to organize an average of 3 events ($\lambda = 3$). If Rachel fails to organize at least 4 events one week, what is the probability that the special guest speaker visited that week?
4. For the events organized last week, if a participant likes each event with a probability of 60%, the expectation of the number of events that a participant likes would be 3.
 - (a) How many events did Rachel organize last week?
 - (b) What is the probability that Rachel did not have a guest speaker visit last week?



Question 14

In this problem, we would like to look into something called a Poisson process, with rate $\lambda > 0$. This is a counting process; for example, you may want to use it to count the number of people arriving at a bus station to wait for the bus, denoted $N(t)$, with respect to time t . There are three conditions:

- (I) $N(0) = 0$, which means you reset the time when the last bus left;
- (II) $N(t)$ has independent increments, which means the number of arrivals depends on the length of the time interval only (i.e., people do not have an App to show when the bus is expected to arrive, and then can rush to the bus station shortly before that);
- (III) The number of arrivals in an interval of length T follows: $N(T) \sim \text{Poisson}(\lambda T)$.

1. Assume that T is very small. With such a small interval, it is safe to assume that the probability of having two or more arrivals is negligible. In other words, either nobody has arrived within this short interval, or only one.

- (a) Show that $\mathbb{P}(N(T) = 0) = e^{-\lambda T} \approx 1 - \lambda T$.

(Hint: Use Taylor series, and claim that any term involving T^2 or higher power is close to zero.)

- (b) Show that $\mathbb{P}(N(T) = 1) \approx \lambda T$.

2. Now we would like to approach the counting process in a different way. Suppose we keep only conditions (I) and (II), and at the same time, $\mathbb{P}(N(T) = 0) = 1 - \lambda T$ and $\mathbb{P}(N(T) = 1) = \lambda T$. Our goal is to show that this will result in condition (III). To do that, for notational simplicity, let us define

$$g_k(t) \triangleq \mathbb{P}(N(t) = k).$$

- (a) Consider the case where $k = 0$. Let Δ be a small time interval. Show that we have

$$g_0(t + \Delta) = g_0(t) \cdot (1 - \lambda \Delta).$$

(Hint: We have no arrival up to time $t + \Delta$, which means that we have no arrival up to time t , AND no arrival for the subsequent time Δ . Try to express this mathematically.)

- (b) Rearranging the terms, we have $\frac{g_0(t + \Delta) - g_0(t)}{\Delta} = -\lambda g_0(t)$. By taking limit of $\Delta \rightarrow 0$, the above becomes an ordinary differential equation. Check that a solution to this is

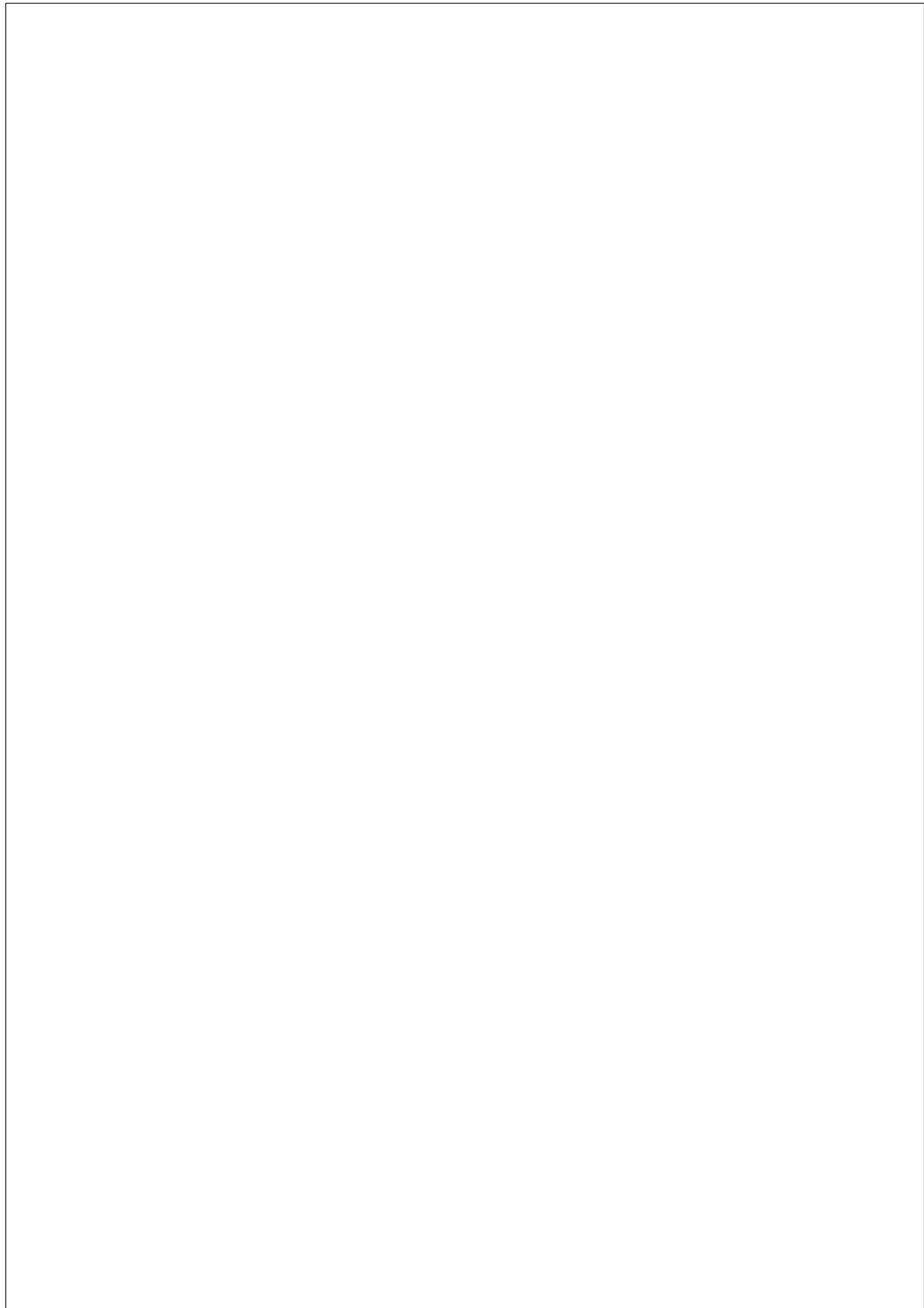
$$g_0(t) = e^{-\lambda t}.$$

- (c) Now consider the other cases where $k \geq 1$. Show that

$$g_k(t + \Delta) = g_k(t)(1 - \lambda \Delta) + g_{k-1}(t)(\lambda \Delta).$$

- (d) Rearranging the terms, we have $\frac{g_k(t + \Delta) - g_k(t)}{\Delta} = -\lambda g_k(t) + \lambda g_{k-1}(t)$. By taking limit of $\Delta \rightarrow 0$, the above becomes an ordinary differential equation. Check that a solution to this is

$$g_k(t) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}.$$



Question 15

Alex, an avid hiker, visits Wildflower Park, renowned for its breathtaking scenery and abundant wildflowers. In one of the open areas, Alex takes the opportunity to observe the wildflowers. Let the discrete random variable X denote the number of wildflowers observed by Alex on a single visit, with possible values 0, 1, 2, and 3 having the probability mass function:

$$\mathbb{P}(X = 0) = 0.1, \quad \mathbb{P}(X = 1) = 0.4, \quad \mathbb{P}(X = 2) = 0.3, \quad \mathbb{P}(X = 3) = 0.2.$$

Answer the following questions:

1. Compute the probability that Alex observes at least two wildflowers on a single visit, i.e., find $\mathbb{P}(X \geq 2)$.
2. Determine the expected value $\mathbb{E}(X)$ and the variance $\text{var}(X)$ of X .
3. Suppose that Alex visits Wildflower Park on 4 independent occasions. Let Y denote the total number of wildflowers observed over these 4 visits. Compute $\mathbb{E}(Y)$ and $\text{var}(Y)$.
4. Compute the probability that Alex observes at least one wildflower on each of the 4 visits.

