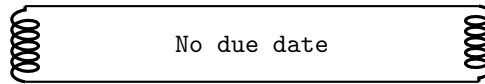


Exercise 1



Instruction: These are for your own practice. There is no need to submit your answer.

Question 1 Assessing risk

In the boardgame called Risk[®], at a given time, a player is the attacker (“A”) and another player is the defender (“D”) of a territory. The outcome of the attack is determined by the following set of rules:

- (I) “A” rolls between one and three dice.
- (II) “D” rolls between one and two dice.
- (III) The highest rolled die from “A” is compared to the highest rolled die from “D”:
 - (a) “A” loses (one combat) if its value is *smaller than or equal to* that from “D”;
 - (b) “A” wins that combat otherwise.
- (IV) The compared dice are discarded, and if both players still have dice left, repeat Step (III) above.

We would like to analyze the odds of winning and losing in different situations.

1. Suppose A rolls one die (e.g., ) and D rolls one die (e.g., ). What is the probability of A succeeding in its attack?
2. Suppose A rolls two dice (e.g., ) and D rolls one die (e.g., ). What is the probability of A succeeding in its attack?
3. Suppose A rolls three dice (e.g., ) and D rolls one die (e.g., ). What is the probability of A succeeding in its attack?
4. Suppose A rolls one die (e.g., ) and D rolls two dice (e.g., ). What is the probability of A succeeding in its attack?
5. (Optional, bonus) Suppose A rolls two dice (e.g., ) and D also rolls two dice (e.g., ).
 - (a) What is the probability of A winning both combats?
 - (b) What is the probability of A winning one combat and losing the other?

1. The situation is simple enough that we could enumerate all $6 \times 6 = 36$ combinations and count the proportion of A winning the attack. Alternatively, we sum the probability by considering each case that D can take. Note that D can be $\{1, 2, 3, 4, 5, 6\}$, each with probability $\frac{1}{6}$. When $D = 6$, it always wins; when $D = 5$, it wins when $A = \{1, 2, 3, 4, 5\}$, i.e., $\frac{5}{6}$ of the time. Continuing with similar arguments, we obtain

$$\mathbb{P}(\text{A wins}) = \sum_{i=1}^5 \mathbb{P}(A > i) \mathbb{P}(D = i) = \frac{5 + 4 + 3 + 2 + 1}{6} \cdot \frac{1}{6} = \frac{15}{36} \approx 41.67\%$$

2. With three dice involved in the combat, we now have $6 \times 6 \times 6 = 216$ combinations to enumerate. Fortunately, the formula above still works, except that $\mathbb{P}(A > i)$ now refers to the larger of the two dice from A. For $i = 1$, A wins unless both dice are one; hence $6^2 - 1^2 = 35$ out of 36 times. Similarly, for $i = 2$, A wins $6^2 - 2^2$ times (except $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$, $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$). Hence,

$$\mathbb{P}(\text{A wins}) = \sum_{i=1}^5 \mathbb{P}(A > i) \mathbb{P}(D = i) = \frac{6^2 \cdot 5 - (1^2 + 2^2 + 3^2 + 4^2 + 5^2)}{6^2} \cdot \frac{1}{6} = \frac{125}{216} \approx 57.87\%$$

3. With four dice involved in the combat, we now have $6 \times 6 \times 6 \times 6 = 1296$ combinations to enumerate. Fortunately, the formula above still works, and that for each i , A wins $6^3 - i^3$ out of 6^3 times. Hence,

$$\mathbb{P}(\text{A wins}) = \sum_{i=1}^5 \mathbb{P}(A > i) \mathbb{P}(D = i) = \frac{6^3 \cdot 5 - (1^3 + 2^3 + 3^3 + 4^3 + 5^3)}{6^3} \cdot \frac{1}{6} = \frac{855}{1296} \approx 65.97\%$$

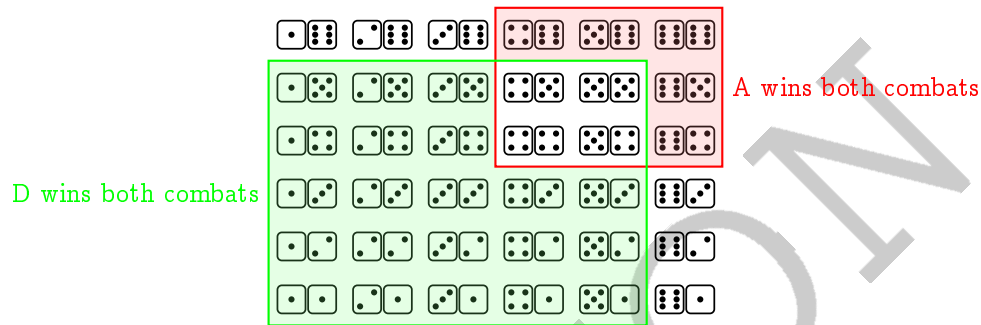
4. With three dice involved in the combat, we now have $6 \times 6 \times 6 = 216$ combinations to enumerate. It is now simpler to loop through different values of A, such that

$$\mathbb{P}(\text{A wins}) = \sum_{i=2}^6 \mathbb{P}(A = i) \mathbb{P}(D < i) = \frac{1}{6} \cdot \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2}{6^2} = \frac{55}{216} \approx 25.46\%$$

5. We need some notations to make the calculation easier. Let A_1 denote the larger value from A's dice roll, and A_2 the smaller value; similarly, D_1 is the bigger and D_2 is the smaller value from D's dice roll.

We count systematically by looping through each value of D_1 ; for each case, we loop through all possible values of D_2 . Then, for each specific $\{D_1, D_2\}$, we enumerate the number of cases (out of $6^4 = 1296$) that A (or D) wins both combats.

Before we do that, let us look at a specific case ($D_1 = \text{3, 4}$, $D_2 = \text{3, 4}$) and consider all 36 combinations of A's dice roll:



The red square consists of all rolls where $A_2 > D_2$, and contains $(6 - D_2)^2 = 3^2$ cases; among them, $(D_1 - D_2)^2 = 2^2$ does not satisfy $A_1 > D_1$, so we have to rule them out. Altogether, only for $3^2 - 2^2 = 5$ cases that we have A winning both combats.

Meanwhile, the green square consists of all rolls where $D_1 \geq A_1$, which amounts to $D_1^2 = 5^2$ cases; we again have to subtract $(D_1 - D_2)^2 = 2^2$ cases where we do not have $D_2 \geq A_2$. Thus, there are $5^2 - 2^2 = 21$ cases that D wins both combats.

$D_1 = 1$:

D_2	occurrence	A wins both	D wins both
1	1	$5^2 - 0^2 = 25$	$1^2 - 0^2 = 1$
total:		25	1

$D_1 = 2$:

D_2	occurrence	A wins both	D wins both
1	2	$5^2 - 1^2 = 24$	$2^2 - 1^2 = 3$
2	1	$4^2 - 0^2 = 16$	$2^2 - 0^2 = 4$
total:		64	10

(Note: There are two cases, 3, 3 and 4, 3 , where $D_1 = 2$ and $D_2 = 1$.)

$D_1 = 3$:

D_2	occurrence	A wins both	D wins both
1	2	$5^2 - 2^2 = 21$	$3^2 - 2^2 = 5$
2	2	$4^2 - 1^2 = 15$	$3^2 - 1^2 = 8$
3	1	$3^2 - 0^2 = 9$	$3^2 - 0^2 = 9$
total:		81	35

$D_1 = 4$:

D_2	occurrence	A wins both	D wins both
1	2	$5^2 - 3^2 = 16$	$4^2 - 3^2 = 7$
2	2	$4^2 - 2^2 = 12$	$4^2 - 2^2 = 12$
3	2	$3^2 - 1^2 = 8$	$4^2 - 1^2 = 15$
4	1	$2^2 - 0^2 = 4$	$4^2 - 0^2 = 16$
<i>total:</i>		76	84

$D_1 = 5$:

D_2	occurrence	A wins both	D wins both
1	2	$5^2 - 4^2 = 9$	$5^2 - 4^2 = 9$
2	2	$4^2 - 3^2 = 7$	$5^2 - 3^2 = 16$
3	2	$3^2 - 2^2 = 5$	$5^2 - 2^2 = 21$
4	2	$2^2 - 1^2 = 3$	$5^2 - 1^2 = 24$
5	1	$1^2 - 0^2 = 1$	$5^2 - 0^2 = 25$
<i>total:</i>		49	165

$D_1 = 6$:

D_2	occurrence	A wins both	D wins both
1	2	$5^2 - 5^2 = 0$	$6^2 - 5^2 = 11$
2	2	$4^2 - 4^2 = 0$	$6^2 - 4^2 = 20$
3	2	$3^2 - 3^2 = 0$	$6^2 - 3^2 = 27$
4	2	$2^2 - 2^2 = 0$	$6^2 - 2^2 = 32$
5	2	$1^2 - 1^2 = 0$	$6^2 - 1^2 = 35$
6	1	$0^2 - 0^2 = 0$	$6^2 - 0^2 = 36$
<i>total:</i>		0	286

So, altogether A wins both combats with probability $\frac{25+64+81+76+49+0}{1296} = \frac{295}{1296} \approx 22.76\%$.

D wins both combats with probability $\frac{1+10+35+84+165+286}{1296} = \frac{581}{1296} \approx 44.83\%$.

The probability that each wins one combat is $1 - \frac{295}{1296} - \frac{581}{1296} = \frac{420}{1296} \approx 32.41\%$.

Question 2

Assume that we have a well-shuffled deck of cards, and we then get five cards among the 52. What is the probability that it is a “full house”? This is defined as having a three-of-a-kind (i.e., the same number or rank, but different suits) and a pair, e.g., you get the nine of hearts, nine of spades, nine of diamonds, king of hearts, king of clubs.

There are $\binom{52}{5} = 2598960$ ways of getting five cards, all equally likely.

Among them, there are 13 possible numbers (called “ranks” or “denominations”) where we can have the three-of-a-kind, and for each number, there are $\binom{4}{3}$ ways of deciding which three suits. Afterwards, there remains 12 possible numbers for the pair, and $\binom{4}{2}$ ways of deciding which two suits. Therefore, the probability of a full house is

$$\mathbb{P}(\text{full house}) = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2598960} \approx 0.14\%$$

Question 3

You are a contestant in a game show, in which there is a prize hidden behind one of three doors with equal likelihood. You are allowed to pick one door. The host, who knows where the prize is, then opens one of the remaining two doors and reveals that there is no prize behind that door. At this point, you have two options:

1. Stick to your original choice.
2. Switch to the other door that has not yet been opened.

Would option 1 give you better odds, or option 2, or they are the same? Explain.

Let us compute the probability of getting the prize with the two options.

1. There is $\frac{1}{3}$ chance that the prize is behind any specific door, so this remains the probability of getting the prize.
2. Let us label the three doors A, B, and C, and assume that you have picked A.
 - (a) If the prize was in A, the host can open B or C, and you will switch and lose.
 - (b) If the prize was in B, the host can only open C, and you will switch to B and win.
 - (c) If the prize was in C, the host can only open B, and you will switch to C and win.

So $\frac{2}{3}$ of the time you get the prize.

Mathematically, $\mathbb{P}(\text{win} | A) \mathbb{P}(A) + \mathbb{P}(\text{win} | B) \mathbb{P}(B) + \mathbb{P}(\text{win} | C) \mathbb{P}(C) = 0 + \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

Thus, option 2 is a better strategy!

Question 4

We have defined “conditional probability” as

$$\mathbb{P}(A|B) \triangleq \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad (1)$$

(assuming $\mathbb{P}(B) \neq 0$) but how do we know that this expression indeed gives a “probability”? Remember from the course notes that there are three axioms that all probability functions must satisfy. Show that $\mathbb{P}(A|B)$ indeed complies with these axioms.

We need to show three things:

1. **Nonnegativity:** Since $\mathbb{P}(A \cap B)$ and $\mathbb{P}(B)$ are both nonnegative, the quantity $\mathbb{P}(A|B)$ defined above will also be nonnegative.
2. **Additivity:** assume two disjoint sets A_1 and A_2 . Consequently, $A_1 \cap B$ and $A_2 \cap B$ are also disjoint, which we will make use of below.

$$\begin{aligned} \mathbb{P}(A_1 \cup A_2 | B) &= \frac{\mathbb{P}((A_1 \cup A_2) \cap B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}((A_1 \cap B) \cup (A_2 \cap B))}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A_1 \cap B)}{\mathbb{P}(B)} + \frac{\mathbb{P}(A_2 \cap B)}{\mathbb{P}(B)} \\ &= \mathbb{P}(A_1 | B) + \mathbb{P}(A_2 | B) \end{aligned}$$

3. **Normalization:**

$$\mathbb{P}(\Omega | B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$$

Question 5

Two students, A and B, toss a biased coin successively, where the probability of head is $\frac{1}{3}$. A tosses first, then B, and then the sequence repeats again starting with A. The person who tosses the first head wins. Find the probability of winning for each.

We first consider the probability that A wins. If A wins in the first round, then the probability is $\frac{1}{3}$. Otherwise, A may win in the third round, given that B does not get a head in the second round. The probability that A wins in the third round is

$$\mathbb{P}(\text{A wins in the third round}) = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{27}.$$

Following the same idea, we can obtain the probability that A wins is

$$\begin{aligned} \mathbb{P}(\text{A wins}) &= \mathbb{P}(\text{A wins in the first round}) + \mathbb{P}(\text{A wins in the third round}) + \dots \\ &= \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^4 + \dots \\ &= \frac{\frac{1}{3}}{1 - \frac{4}{9}} = \frac{3}{5}. \end{aligned}$$

Probability that B wins would be $1 - \frac{3}{5} = \frac{2}{5}$.

Question 6

In a game, the players need to control a game character to defeat as many monsters as possible. The players will only encounter one monster each round. A player's chance of defeating a monster in each round is p . If the player fails to defeat the monster for r rounds in total, the game ends.

1. Denote by X the number of rounds in which this player successfully defeats the monster before the game ends. What is the PMF of X ?
2. What is the expected value of X ?

1. Suppose that when the game ends, the player has defeated the monster for k rounds, and fails to do so for r rounds, so that in total, the person has played $k + r$ rounds. Note that in the last round, the player must fail to defeat the monster. In the preceding $k + r - 1$ rounds, the player can be defeating the monster in any of the k rounds. Therefore,

$$\begin{aligned} p_X(X = k) &= \binom{k+r-1}{k} p^k (1-p)^{(k+r-1)-k} \cdot (1-p) \\ &= \binom{k+r-1}{k} p^k (1-p)^r \quad \text{where } k = 0, 1, \dots \end{aligned}$$

2. We make use of the usual formula

$$\begin{aligned} \mathbb{E}(X) &= \sum_{k=0}^{\infty} k \cdot p_X(X = k) \\ &= \sum_{k=0}^{\infty} k \cdot \binom{k+r-1}{k} p^k (1-p)^r \\ &= \sum_{k=1}^{\infty} \frac{(k+r-1)!}{(k-1)!(r-1)!} p^k (1-p)^r \\ &= \frac{rp}{1-p} \sum_{k=1}^{\infty} \frac{(k+r-1)!}{(k-1)!r!} p^{k-1} (1-p)^{r+1} \\ &= \frac{rp}{1-p} \sum_{k=1}^{\infty} \binom{r+k-1}{k-1} p^{k-1} (1-p)^{r+1} \\ &= \frac{rp}{1-p} \sum_{z=0}^{\infty} \binom{z+r'-1}{z} p^z (1-p)^{r'} \\ &= \frac{rp}{1-p} \end{aligned}$$

where we have let $z = k - 1$ and $r' = r + 1$. The reason why we know the summation is one is because if you compare the quantity after the summation with the solution for $p_X(X = k)$ above, you can see that we are essentially summing the probabilities for all cases of z . Note that such a sum must equal to one, by the axiom of probability.

Question 7

Let X be a discrete random variable that takes on the integer values from 0 to 9 with equal probability. Also, the notation $r = a \pmod{b}$ means r is the remainder when a is divided by b .

1. Find the PMF of Y , where $Y = X \pmod{3}$.
2. Find the PMF of Z , where $Z = 5 \pmod{(X+1)}$.

1. Using the formula $p_Y(y) = \sum_{\{x|x \pmod{3}=y\}} p_X(x)$, we obtain

$$p_Y(0) = p_X(0) + p_X(3) + p_X(6) + p_X(9) = 4/10$$

$$p_Y(1) = p_X(1) + p_X(4) + p_X(7) = 3/10$$

$$p_Y(2) = p_X(2) + p_X(5) + p_X(8) = 3/10$$

$$p_Y(y) = 0, \quad \text{if } y \notin \{0, 1, 2\}.$$

2. Similarly, using the formula $p_Z(z) = \sum_{\{x|5 \pmod{(x+1)}=z\}} p_X(x)$, we obtain

$$p_Z(z) = \begin{cases} 2/10, & \text{if } z = 0 \\ 2/10, & \text{if } z = 1 \\ 1/10, & \text{if } z = 2 \\ 5/10, & \text{if } z = 5 \\ 0, & \text{otherwise} \end{cases}$$

Question 8

The random variable X has the following PMF

$$p_X(x) = \begin{cases} |x|/a & \text{if } x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

1. Find a and $\mathbb{E}(X)$.
2. What is the PMF of the random variable $Z = (X - \mathbb{E}(X))^2$?
3. Using the result above, find the variance of X .
4. Find the variance of X using the formula $\text{var}(X) = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$?

1. The scalar a must satisfy

$$\sum_x p_X(x) = \frac{1}{a} \sum_{x=-3}^3 |x| = 1$$

so

$$a = \sum_{x=-3}^3 |x| = |-3| + |-2| + |-1| + |1| + |2| + |3| = 12$$

We also have $\mathbb{E}(X) = 0$ because the PMF is symmetric around 0.

2. If $z \in \{1, 4, 9\}$, then

$$p_Z(z) = p_X(\sqrt{z}) + p_X(-\sqrt{z}) = \frac{\sqrt{z}}{12} + \frac{\sqrt{z}}{12} = \frac{\sqrt{z}}{6}$$

Otherwise $p_Z(z) = 0$.

$$3. \text{ var}(X) = \mathbb{E}(Z) = \sum_z z p_Z(z) = \sum_{z \in \{1, 4, 9\}} \frac{z^{\frac{3}{2}}}{6} = 6.$$

4. We have

$$\begin{aligned} \text{var}(X) &= \sum_x (x - \mathbb{E}(X))^2 p_X(x) \\ &= 1^2 \cdot (p_X(-1) + p_X(1)) + 2^2 \cdot (p_X(-2) + p_X(2)) + 3^2 \cdot (p_X(-3) + p_X(3)) \\ &= 2 \cdot \frac{1}{12} + 8 \cdot \frac{2}{12} + 18 \cdot \frac{3}{12} \\ &= 6 \end{aligned}$$

Question 9

Suppose that X and Y are independent geometric random variables, both with parameter p . Show that

$$\mathbb{P}(X = i | X + Y = n) = \frac{1}{n-1} \quad i = 1, \dots, n-1.$$

Note that this is a discrete uniform distribution, which has the same probability as long as i is within the prescribed range.

We can express

$$\mathbb{P}(X = i | X + Y = n) = \frac{\mathbb{P}(X = i \cap X + Y = n)}{\mathbb{P}(X + Y = n)} = \frac{\mathbb{P}(X = i) \mathbb{P}(Y = n - i)}{\mathbb{P}(X + Y = n)}$$

where we have made use of independence of X and Y . One interpretation is to take X as the number of coin flips until the first appearance of head, and Y as the additional number of coin flips until the second appearance of head.

Now, note that

$$\begin{aligned} \mathbb{P}(X = i) &= (1-p)^{i-1}p & i \geq 1 \\ \mathbb{P}(Y = n-i) &= (1-p)^{n-i-1}p & n-i \geq 1 \end{aligned}$$

and therefore

$$\mathbb{P}(X = i) \mathbb{P}(Y = n - i) = \begin{cases} (1-p)^{n-2}p^2 & i = 1, \dots, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that this expression is independent of i as long as it is within the range of $[1, n-1]$. The denominator, $\mathbb{P}(X + Y = n)$, is also independent of i . Thus, the probability must be the same for all values of i within the prescribed range, which has $n-1$ elements. Hence, we can conclude that

$$\mathbb{P}(X = i | X + Y = n) = \frac{1}{n-1} \quad i = 1, \dots, n-1$$

Question 10

The number of crossways a taxi passes through in a day follows a Poisson distribution, i.e., $Y \sim \text{Poisson}(\lambda)$. Assume that the traffic lights at each crossway work independently and the probability of encountering a red light at each crossway is $p > 0$.

1. If a taxi passes through k crossways in a day, what is the distribution of the number of red lights it encounters?
2. What is the distribution of the number of red lights a taxi encounters in one day?

Let the number of red lights the taxi encounters be variable X .

1. Knowing that the taxi passes through k crossways and that each is independent, we have a succession of Bernoulli trials, i.e., this is a binomial distribution.

$$\mathbb{P}(X = x | Y = k) = \binom{k}{x} p^x (1-p)^{k-x}, \quad \text{where } x = 0, 1, \dots, k$$

2. Now we do not fix the number of crossways the taxi passes. Therefore, we need to add all possible numbers, starting from $k = x$ (the number of crossways must be at least the same as the number of red lights).

$$\begin{aligned} \mathbb{P}(X = x) &= \sum_{k=x}^{\infty} \mathbb{P}(Y = k) \mathbb{P}(X = x | Y = k) \\ &= \sum_{k=x}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \binom{k}{x} p^x (1-p)^{k-x} \\ &= \sum_{k=x}^{\infty} \frac{(\lambda(1-p))^{k-x}}{x!(k-x)!} e^{-\lambda} (\lambda p)^x \\ &= \sum_{k=x}^{\infty} \frac{(\lambda(1-p))^{k-x}}{x!(k-x)!} e^{-\lambda} (\lambda p)^x \\ &= \frac{(\lambda p)^x}{x!} e^{-\lambda} \sum_{j=0}^{\infty} \frac{(\lambda(1-p))^j}{j!} \\ &= \frac{(\lambda p)^x}{x!} e^{-\lambda} e^{\lambda(1-p)} \\ &= \frac{(\lambda p)^x}{x!} e^{-\lambda p} \quad \text{where } x = 0, 1, \dots, k \end{aligned}$$

So X also follows the Poisson distribution with parameter λp .

Question 11

A person goes to a store to buy cups. After collecting 20 cups with a box, he wants to verify their quality. Rather than checking each cup individually, he plans to randomly examine 4 cups.

1. Find the probability of there being no quality problem with the 4 selected cups under different assumptions as follows:
 - (a) Assume that the probability of each cup in the box being with quality problem is 0.1.
 - (b) Assume that 2 cups in this box are with quality problems.
 - (c) Assume that the probabilities of 0, 1, and 2 cups in this box having quality problems are 0.8, 0.1, and 0.1, respectively.
2. Given that the person has checked 4 cups and found no problem, find the probability that the other 16 cups are all in good quality under different assumptions as follows:
 - (a) Assume that the probability of each cup in the box being with quality problem is 0.1.
 - (b) Assume that the probabilities of 0, 1, and 2 cups in this box having quality problems are 0.8, 0.1, and 0.1, respectively.

1. Let A denote the event of there being no quality problem with the 4 cups to be checked.

(a) If the probability of each cup having quality problem is 0.1, there is

$$\mathbb{P}(A) = (1 - 0.1)^4 = 0.6561.$$

(b) If 2 cups in the box have quality problems, the probability that they are not selected and checked is

$$\mathbb{P}(A) = \frac{\binom{18}{4}}{\binom{20}{4}} = \frac{12}{19} \approx 0.63.$$

Another way to calculate it is to consider picking the cups sequentially, so

$$\mathbb{P}(A) = \frac{18}{20} \cdot \frac{17}{19} \cdot \frac{16}{18} \cdot \frac{15}{17} = \frac{12}{19}$$

(c) Let B_n denote the event of n cups in the box have quality problems. According to the assumption, there are

$$\mathbb{P}(B_0) = 0.8, \quad \mathbb{P}(B_1) = 0.1, \quad \mathbb{P}(B_2) = 0.1.$$

Similar to the answer to (b), we can calculate the conditional probabilities as follows:

$$\mathbb{P}(A | B_0) = 1, \quad \mathbb{P}(A | B_1) = \frac{\binom{19}{4}}{\binom{20}{4}} = \frac{4}{5}, \quad \mathbb{P}(A | B_2) = \frac{\binom{18}{4}}{\binom{20}{4}} = \frac{12}{19}.$$

Using the total probability theorem,

$$\mathbb{P}(A) = \sum_{n=0}^2 \mathbb{P}(A | B_n) \mathbb{P}(B_n) \approx 0.94.$$

2. Let C denote the event of there being no quality problem with the other 16 cups that are not checked.

(a) Under this assumption, events A and C are independent. Using the property of independence, the probability of $\mathbb{P}(C | A)$ is

$$\mathbb{P}(C | A) = \mathbb{P}(C) = (1 - 0.1)^{16} \approx 0.1853.$$

(b) Given that 4 cups to be checked have no quality problem, the claim that the other 16 cups also have no problem is equivalent to the claim that 0 cups in the box has quality problem. As such, using Bayes' rule,

$$\mathbb{P}(C | A) = \mathbb{P}(B_0 | A) = \frac{\mathbb{P}(B_0) \mathbb{P}(A | B_0)}{\mathbb{P}(A)} \approx \frac{0.8 \times 1}{0.94} \approx 0.85.$$

Question 12

There is a room with 4 windows of the same size, only one of which is open. A pigeon flies into the room through the open window and can only fly away through the open window. Now the pigeon flies around the room, trying to escape. Assume that the pigeon has no memory and flies randomly towards each window.

1. Let X denote the number of attempts the pigeon makes to escape the room.
 - (a) Successfully choosing the open window or not is a Bernoulli trial. What is the probability of the Bernoulli trial? What kind of distribution does X follow?
 - (b) Let k denote the number of needed trials. Express the probability mass function (PMF) of X .
2. The owner claims that his pigeon has memory and will not try the same window more than once. Let Y denote the number of attempts this smart pigeon makes to escape the room.
 - (a) Find all possible values of Y and explain the situations for each one.
 - (b) Let i denote the number of needed trials. Express the PMF of Y .
3.
 - (a) Find the probability that the number of attempts X is less than Y (i.e., $\mathbb{P}(X < Y)$).
 - (b) Find the probability that Y is less than X (i.e., $\mathbb{P}(Y < X)$).

1. Probability Distribution of X

- (a) The pigeon has no memory and flies randomly towards each window. The probability of choosing the open window is $\frac{1}{4}$, and the probability of choosing a closed window is $\frac{3}{4}$. The number of attempts X follows a geometric distribution with success probability $p = \frac{1}{4}$.

- (b) The PMF of X is:

$$p_X(k) = \left(\frac{3}{4}\right)^{k-1} \left(\frac{1}{4}\right), \quad k = 1, 2, 3, \dots$$

2. Probability Distribution of Y

- (a) The pigeon has memory and will not try the same window more than once. The possible values of Y are 1, 2, 3, and 4.

$Y = 1$: The pigeon chooses the open window on the first attempt.

$Y = 2$: The pigeon chooses a closed window on the first attempt and an open window on the second attempt.

$Y = 3$: The pigeon chooses two closed windows on the first two attempts and the open window on the third attempt.

$Y = 4$: The pigeon chooses all three closed windows on the first three attempts and the open window on the fourth attempt.

- (b)

$$\mathbb{P}(Y = 1) = \frac{1}{4}$$

$$\mathbb{P}(Y = 2) = \left(\frac{3}{4}\right) \left(\frac{1}{3}\right) = \frac{1}{4}$$

$$\mathbb{P}(Y = 3) = \left(\frac{3}{4}\right) \left(\frac{2}{3}\right) \left(\frac{1}{2}\right) = \frac{1}{4}$$

$$\mathbb{P}(Y = 4) = \left(\frac{3}{4}\right) \left(\frac{2}{3}\right) \left(\frac{1}{2}\right) (1) = \frac{1}{4}$$

Thus, the PMF of Y is:

$$p_Y(i) = \frac{1}{4}, \quad i = 1, 2, 3, 4$$

3. Probabilities $\mathbb{P}(X < Y)$ and $\mathbb{P}(Y < X)$

Since X and Y are independent, we can calculate the joint probabilities.

(a)

$$\begin{aligned}\mathbb{P}(X < Y) &= \sum_{i=1}^3 \sum_{j=i+1}^4 \mathbb{P}((X = i) \cap (Y = j)) \\ &= \sum_{i=1}^3 \sum_{j=i+1}^4 \mathbb{P}(X = i) \mathbb{P}(Y = j)\end{aligned}$$

Substituting the PMFs:

$$\begin{aligned}\mathbb{P}(X < Y) &= \sum_{i=1}^3 \left(\frac{3}{4}\right)^{i-1} \left(\frac{1}{4}\right) \sum_{j=i+1}^4 \frac{1}{4} \\ &= \sum_{i=1}^3 \left(\frac{3}{4}\right)^{i-1} \left(\frac{1}{4}\right) \left(\frac{4-i}{4}\right) \\ &= \left(\frac{1}{4}\right) \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right) \left(\frac{1}{4}\right) \left(\frac{2}{4}\right) + \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right) \left(\frac{1}{4}\right) \\ &= \frac{81}{256} \approx 0.316\end{aligned}$$

(b) The probability that X is equal to Y is:

$$\begin{aligned}\mathbb{P}(X = Y) &= \sum_{k=1}^4 \mathbb{P}((X = k) \cap (Y = k)) \\ &= \sum_{k=1}^4 \mathbb{P}(X = k) \mathbb{P}(Y = k) \\ &= \sum_{k=1}^4 \left(\frac{3}{4}\right)^{k-1} \left(\frac{1}{4}\right) \left(\frac{1}{4}\right) \\ &= \frac{175}{1024}\end{aligned}$$

So that

$$\begin{aligned}\mathbb{P}(Y < X) &= 1 - \mathbb{P}(X < Y) - \mathbb{P}(X = Y) \\ &= 1 - \frac{81}{256} - \frac{175}{1024} \\ &= \frac{525}{1024} \approx 0.513\end{aligned}$$

Question 13

Every week, Rachel is tasked with organizing 4 community events. Let X represent the number of events she successfully organizes each week, which follows a Poisson distribution with $\lambda = 6$.

1. What is the probability that Rachel successfully organizes enough events for a week?
2. Let Y denote the participants' boredom index for the week. The relationship between Y and X is given by: $Y = 1/(X + 1)$. Calculate the expectation of Y .
3. Unfortunately, each week there is a 40% chance that a special guest speaker is visiting, and during those weeks Rachel is only able to organize an average of 3 events ($\lambda = 3$). If Rachel fails to organize at least 4 events one week, what is the probability that the special guest speaker visited that week?
4. For the events organized last week, if a participant likes each event with a probability of 60%, the expectation of the number of events that a participant likes would be 3.
 - (a) How many events did Rachel organize last week?
 - (b) What is the probability that Rachel did not have a guest speaker visit last week?

1. Let X denote the number of community events organized by Rachel within a week,

$$\mathbb{P}(X \geq 4) = 1 - \mathbb{P}(X \leq 3) = 1 - \frac{6^0 \times e^{-6}}{0!} - \frac{6^1 \times e^{-6}}{1!} - \frac{6^2 \times e^{-6}}{2!} - \frac{6^3 \times e^{-6}}{3!} = 0.8488.$$

2. According to the example related to the Poisson random variable from Lecture 2,

$$\mathbb{E}(Y) = \mathbb{E}(1/(1+X)) = \frac{1}{\lambda} (1 - e^{-\lambda}) = 0.1662$$

3. Let S denote the presence of a special guest speaker, where $S = 0$ indicates that the special guest is absent and $S = 1$ indicates that the special guest is present.

$$\mathbb{P}(S = 0) = 0.6$$

$$\mathbb{P}(S = 1) = 0.4$$

$$\mathbb{P}(X < 4 | S = 0) = \frac{6^0 \times e^{-6}}{0!} + \frac{6^1 \times e^{-6}}{1!} + \frac{6^2 \times e^{-6}}{2!} + \frac{6^3 \times e^{-6}}{3!} = 0.1512$$

$$\mathbb{P}(X < 4 | S = 1) = \frac{3^0 \times e^{-3}}{0!} + \frac{3^1 \times e^{-3}}{1!} + \frac{3^2 \times e^{-3}}{2!} + \frac{3^3 \times e^{-3}}{3!} = 0.6472$$

According to Bayes' rule:

$$\begin{aligned} \mathbb{P}(S = 1 | X < 4) &= \frac{\mathbb{P}(S = 1) \times \mathbb{P}(X < 4 | S = 1)}{\mathbb{P}(S = 1) \times \mathbb{P}(X < 4 | S = 1) + \mathbb{P}(S = 0) \times \mathbb{P}(X < 4 | S = 0)} \\ &= \frac{0.40 \times 0.6472}{0.4 \times 0.6472 + 0.6 \times 0.1512} = 0.7405 \end{aligned}$$

4. Let n denote the number of community events organized by Rachel last week, and Q denote the binomial random variable representing the number of events that the participant likes, with PMF

$$p_Q(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, \dots, n$$

- (a) The expectation of Q is calculated as:

$$\mathbb{E}(Q) = np = 3 \quad p = 0.6, \quad \therefore n = 5$$

- (b)

$$\mathbb{P}(X = 5 | S = 1) = \frac{3^5 \times e^{-3}}{5!} = 0.1008$$

$$\mathbb{P}(X = 5 | S = 0) = \frac{6^5 \times e^{-6}}{5!} = 0.1606$$

$$\mathbb{P}(S = 0 | X = 5) = \frac{\mathbb{P}(S = 0) \times \mathbb{P}(X = 5 | S = 0)}{\mathbb{P}(S = 1) \times \mathbb{P}(X = 5 | S = 1) + \mathbb{P}(S = 0) \times \mathbb{P}(X = 5 | S = 0)} = 0.7050$$

Question 14

In this problem, we would like to look into something called a Poisson process, with rate $\lambda > 0$. This is a counting process; for example, you may want to use it to count the number of people arriving at a bus station to wait for the bus, denoted $N(t)$, with respect to time t . There are three conditions:

- (I) $N(0) = 0$, which means you reset the time when the last bus left;
- (II) $N(t)$ has independent increments, which means the number of arrivals depends on the length of the time interval only (i.e., people do not have an App to show when the bus is expected to arrive, and then can rush to the bus station shortly before that);
- (III) The number of arrivals in an interval of length T follows: $N(T) \sim \text{Poisson}(\lambda T)$.

1. Assume that T is very small. With such a small interval, it is safe to assume that the probability of having two or more arrivals is negligible. In other words, either nobody has arrived within this short interval, or only one.

- (a) Show that $\mathbb{P}(N(T) = 0) = e^{-\lambda T} \approx 1 - \lambda T$.
(Hint: Use Taylor series, and claim that any term involving T^2 or higher power is close to zero.)

- (b) Show that $\mathbb{P}(N(T) = 1) \approx \lambda T$.

2. Now we would like to approach the counting process in a different way. Suppose we keep only conditions (I) and (II), and at the same time, $\mathbb{P}(N(T) = 0) = 1 - \lambda T$ and $\mathbb{P}(N(T) = 1) = \lambda T$. Our goal is to show that this will result in condition (III). To do that, for notational simplicity, let us define

$$g_k(t) \triangleq \mathbb{P}(N(t) = k).$$

- (a) Consider the case where $k = 0$. Let Δ be a small time interval. Show that we have

$$g_0(t + \Delta) = g_0(t) \cdot (1 - \lambda \Delta).$$

(Hint: We have no arrival up to time $t + \Delta$, which means that we have no arrival up to time t , AND no arrival for the subsequent time Δ . Try to express this mathematically.)

- (b) Rearranging the terms, we have $\frac{g_0(t + \Delta) - g_0(t)}{\Delta} = -\lambda g_0(t)$. By taking limit of $\Delta \rightarrow 0$, the above becomes an ordinary differential equation. Check that a solution to this is

$$g_0(t) = e^{-\lambda t}.$$

- (c) Now consider the other cases where $k \geq 1$. Show that

$$g_k(t + \Delta) = g_k(t)(1 - \lambda \Delta) + g_{k-1}(t)(\lambda \Delta).$$

- (d) Rearranging the terms, we have $\frac{g_k(t + \Delta) - g_k(t)}{\Delta} = -\lambda g_k(t) + \lambda g_{k-1}(t)$. By taking limit of $\Delta \rightarrow 0$, the above becomes an ordinary differential equation. Check that a solution to this is

$$g_k(t) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}.$$

1. Since $N(T) \sim \text{Poisson}(\lambda T)$, we have:

(a) $\mathbb{P}(N(T) = 0) = \frac{e^{-(\lambda T)}(\lambda T)^0}{0!} = e^{-\lambda T}$. Now using Taylor series,

$$\mathbb{P}(N(T) = 0) \approx 1 + \frac{(-\lambda T)}{1!} + \frac{(-\lambda T)^2}{2!} + \dots = 1 - \lambda T$$

(b) Since $N(T)$ can only be zero or one within a small interval,

$$\mathbb{P}(N(T) = 1) \approx 1 - \mathbb{P}(N(T) = 0) = \lambda T.$$

2. From condition (II), the probability that there is no arrival from t to $t + \Delta$ is the same as from 0 to Δ , which equals to $1 - \lambda \Delta$.

(a) As the hint says, no arrival up to time $t + \Delta$ means that we have no arrival up to time t AND no arrival for the subsequent time Δ , i.e.,

$$\underbrace{g_0(t + \Delta)}_{\text{no arrival } t + \Delta} = \underbrace{g_0(t)}_{\text{no arrival } t} \cdot \underbrace{(1 - \lambda \Delta)}_{\text{no arrival } \Delta}.$$

(b) Taking limit, we can write the differential equation as $\frac{d}{dt}g_0(t) = -\lambda g_0(t)$. The general solution is

$$g_0(t) = Ae^{-\lambda t}.$$

Since $g_0(0) = 1$, we must have $A = 1$.

(c) The probability that there are k arrivals by time $t + \Delta$ is, by the principle of total probability, the sum of k arrivals by time t and no arrival in the subsequent Δ , AND $k - 1$ arrivals by time t and one arrival in the subsequent Δ . Therefore,

$$g_k(t + \Delta) = g_k(t)(1 - \lambda \Delta) + g_{k-1}(t)(\lambda \Delta).$$

(d) Taking limit, we can write the differential equation as $\frac{d}{dt}g_k(t) = -\lambda [g_k(t) - g_{k-1}(t)]$. Let us now verify the form given in the problem:

$$\begin{aligned} \frac{d}{dt}g_k(t) &= \frac{d}{dt} \left[\frac{e^{-\lambda t}(\lambda t)^k}{k!} \right] \\ &= -\lambda \frac{e^{-\lambda t}(\lambda t)^k}{k!} + k \frac{e^{-\lambda t}(\lambda t)^{k-1}}{k!} \\ &= -\lambda g_k(t) + \lambda \frac{e^{-\lambda t}(\lambda^{k-1} t^{k-1})}{(k-1)!} \\ &= -\lambda [g_k(t) - g_{k-1}(t)] \end{aligned}$$

For a similar discussion, see “Origin of the Poisson random variable” in Section 3.5 of the supplementary textbook.

Question 15

Alex, an avid hiker, visits Wildflower Park, renowned for its breathtaking scenery and abundant wildflowers. In one of the open areas, Alex takes the opportunity to observe the wildflowers. Let the discrete random variable X denote the number of wildflowers observed by Alex on a single visit, with possible values 0, 1, 2, and 3 having the probability mass function:

$$\mathbb{P}(X = 0) = 0.1, \quad \mathbb{P}(X = 1) = 0.4, \quad \mathbb{P}(X = 2) = 0.3, \quad \mathbb{P}(X = 3) = 0.2.$$

Answer the following questions:

1. Compute the probability that Alex observes at least two wildflowers on a single visit, i.e., find $\mathbb{P}(X \geq 2)$.
2. Determine the expected value $\mathbb{E}(X)$ and the variance $\text{var}(X)$ of X .
3. Suppose that Alex visits Wildflower Park on 4 independent occasions. Let Y denote the total number of wildflowers observed over these 4 visits. Compute $\mathbb{E}(Y)$ and $\text{var}(Y)$.
4. Compute the probability that Alex observes at least one wildflower on each of the 4 visits.

1. Let A denote the event that Alex observes at least two wildflowers on a single visit. Then,

$$\mathbb{P}(X \geq 2) = \mathbb{P}(X = 2) + \mathbb{P}(X = 3) = 0.3 + 0.2 = 0.5.$$

2. The expected value of X is

$$\mathbb{E}(X) = \sum_{x=0}^3 x \mathbb{P}(X = x) = 0 \cdot 0.1 + 1 \cdot 0.4 + 2 \cdot 0.3 + 3 \cdot 0.2 = 1.6.$$

To compute the variance, first find the second moment:

$$\mathbb{E}(X^2) = \sum_{x=0}^3 x^2 \mathbb{P}(X = x) = 0^2 \cdot 0.1 + 1^2 \cdot 0.4 + 2^2 \cdot 0.3 + 3^2 \cdot 0.2 = 0 + 0.4 + 1.2 + 1.8 = 3.4.$$

Hence, the variance is

$$\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3.4 - (1.6)^2 = 3.4 - 2.56 = 0.84.$$

3. Suppose Alex visits Wildflower Park on 4 independent occasions. Let X_1, X_2, X_3, X_4 be the number of wildflowers observed on each visit (each with the same distribution as X), and define

$$Y = X_1 + X_2 + X_3 + X_4.$$

By the linearity of expectation,

$$\mathbb{E}(Y) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4) = 4\mathbb{E}(X) = 4 \times 1.6 = 6.4.$$

Since the visits are independent, the variance of the sum is the sum of the variances:

$$\text{var}(Y) = \text{var}(X_1) + \text{var}(X_2) + \text{var}(X_3) + \text{var}(X_4) = 4\text{var}(X) = 4 \times 0.84 = 3.36.$$

4. The probability that Alex observes at least one wildflower on a single visit is

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - 0.1 = 0.9.$$

Since the visits are independent, the probability that he observes at least one wildflower on each of the 4 visits is

$$(\mathbb{P}(X \geq 1))^4 = (0.9)^4 \approx 0.6561.$$