Continuous Random Variables

Prof. Edmund Lam

Department of Electrical and Electronic Engineering
The University of Hong Kong

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https://www.eee.hku.hk/~elec2844

Random variables

Continuous random variable X:

- has a nonnegative function

$$\boxed{f_X(x) \geqslant 0} \tag{1}$$

  for all x, called probability density function (PDF)
- Integrating ("continuous summing") this PDF gives us the probability of an event:

$$\boxed{\mathbb{P}(X \in B) = \int_B f_X(x)\,dx} \tag{2}$$

  for every subset B of the entire space

Random variables

Continuous random variable X:

- When B includes "everything" (Ω), then

$$\boxed{\mathbb{P}(X \in \Omega) = \int_\Omega f_X(x)\,dx = 1} \tag{3}$$

- For B to refer to a range $[a, b]$,

$$\boxed{\mathbb{P}(a \leqslant X \leqslant b) = \int_a^b f_X(x)\,dx} \tag{4}$$

- Integral is zero when the range contains only a single number, so

$$\mathbb{P}(a \leqslant X \leqslant B) = \mathbb{P}(a < X < b) = \mathbb{P}(a \leqslant X < b) = \mathbb{P}(a < X \leqslant b)$$

Random variables

Continuous random variable X:

- Normalization:

$$\boxed{\mathbb{P}(-\infty < X < \infty) = \int_{-\infty}^{\infty} f_X(x)\,dx = 1} \tag{5}$$

- For an interval $[x, x + \delta]$:

$$\mathbb{P}([x, x+\delta]) = \int_x^{x+\delta} f_X(t)\,dt \approx f_X(x) \cdot \delta$$

  So $f_X(x)$ is the "probability mass per unit length" near x
- Since $f_X(x)$ is not the probability of any event, it does not have to be $\leqslant 1$.

Example

Is $f_X(x)$ a PDF?

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}} & 0 < x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

ANS: Even though $f_X(x)$ becomes infinitely large as $x \to 0$, we still have $f_X(x) \geqslant 0$ for all x, and

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}dx = \sqrt{x}\Big|_0^1 = 1$$

So this is a PDF.

Example

Let $f_X(x) = \begin{cases} x^2 & |x| < a \\ 0 & \text{otherwise.} \end{cases}$ be a PDF. Find a.

ANS: We integrate over the entire space, and set it to one:

$$\int_{-a}^{a} x^2 dx = \frac{x^3}{3}\Big|_{-a}^{a} = \frac{2a^3}{3} = 1$$

Therefore, $a = \sqrt[3]{\frac{3}{2}}$.

1 Expectation, mean, variance
2 Common continuous random variables
3 Cumulative density function
4 Derived distributions
5 Joint PDF, conditioning, independence
6 Continuous Bayes' rule

Expectation

The expectation or expected value or mean of a continuous r.v. X is

$$\boxed{\mathbb{E}(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx} \tag{6}$$

Similar to the discrete case, it is a very useful number as a "representative" value of X, and is also viewed as the "center of gravity."

Note: There can be problematic cases with integrals: even if $f_X(x)$ is symmetric about 0, the expected value is in fact not defined, e.g. $f_X(x) = 1/\pi(1+x^2)$

Expectation and variance

The formulas for continuous r.v. are similar to those for discrete:

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \tag{7}$$

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \tag{8}$$

$$\text{var}(X) = \mathbb{E}\Big(\big(X - \mathbb{E}(X)\big)^2\Big) = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 f_X(x)\, dx \tag{9}$$

$$\text{var}(X) = \mathbb{E}(X^2) - \big(\mathbb{E}(X)\big)^2 \geqslant 0 \tag{10}$$

$$\mathbb{E}(Y) = a\mathbb{E}(X) + b \tag{11}$$

$$\text{var}(Y) = a^2 \text{var}(X) \tag{12}$$

where $Y = aX + b$

1. Expectation, mean, variance
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Continuous random variable

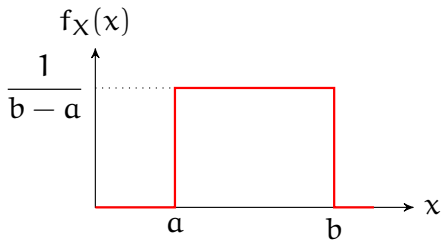
It is useful to know the mean and variance of some common distributions. A summary:

	mean: $\mathbb{E}(X)$	variance: $\text{var}(X)$
uniform: $X \sim U(a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
exponential: $X \sim \text{Exp}(\lambda)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
normal: $X \sim \mathcal{N}(\mu, \sigma^2)$	μ	σ^2

(1) Uniform random variable

Continuous uniform random variable over a range $[a, b]$ has PDF

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \leqslant x \leqslant b \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$



(1) Uniform random variable

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \leqslant x \leqslant b \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^b x \cdot \tfrac{1}{b-a}\,dx = \tfrac{1}{b-a} \cdot \tfrac{1}{2}x^2\Big|_a^b = \tfrac{a+b}{2}$$

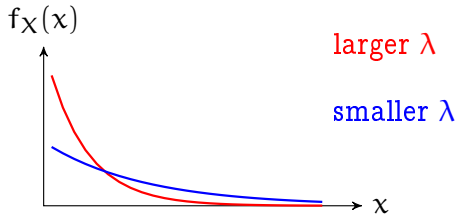
$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_a^b x^2 \cdot \tfrac{1}{b-a}\,dx = \tfrac{1}{b-a} \cdot \tfrac{1}{3}x^3\Big|_a^b = \tfrac{a^2+ab+b^2}{3}$$

$$\mathrm{var}(X) = \tfrac{a^2+ab+b^2}{3} - \left(\tfrac{a+b}{2}\right)^2 = \tfrac{(b-a)^2}{12}$$

(2) Exponential random variable

Exponential random variable with parameter $\lambda > 0$ has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geqslant 0 \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$



(2) Exponential random variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}(X) = \int_0^{\infty} x\lambda e^{-\lambda x}\,dx = (-xe^{-\lambda x})\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx = 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^{\infty} = \frac{1}{\lambda}$$

$$\mathbb{E}(X^2) = \int_0^{\infty} x^2\lambda e^{-\lambda x}\,dx = (-x^2e^{-\lambda x})\Big|_0^{\infty} + \int_0^{\infty} 2xe^{-\lambda x}\,dx = \frac{2}{\lambda}\mathbb{E}(X) = \frac{2}{\lambda^2}$$

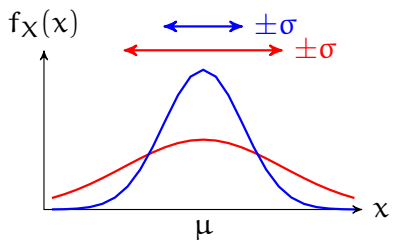
$$\mathrm{var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

(3) Normal random variable

Gaussian or normal r.v. with parameters $(\mu, \sigma > 0)$ has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \tag{15}$$

This is probably the most common distribution of all (with good scientific reason)!



(3) Normal random variable

Show that the normal distribution given by Eq. (15) has unit area.

Proof: Let $I = \int_{-\infty}^{\infty} e^{-x^2/2} dx$. The trick is to go two-dimensional by computing I^2.

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy$$

Then, we convert to polar coordinates:

$$\begin{aligned} I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r dr d\theta &= 2\pi \int_0^{\infty} e^{-r^2/2} r dr \\ &= 2\pi \int_0^{\infty} e^{-u} du = 2\pi(-e^{-u})\Big|_0^{\infty} = 2\pi \\ I &= \sqrt{2\pi} \end{aligned}$$

(3) Normal random variable

Thus, we have shown that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = 1$$

Then, we make the change of variable $s = \dfrac{x-\mu}{\sigma}$:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = 1$$

since $ds = dx/\sigma$.

(3) Normal random variable

Normal distribution $X \sim \mathcal{N}(\mu, \sigma^2)$ has mean μ and variance σ^2.

Proof: For the mean, we will appeal to the fact that it is symmetric around μ and that the integral is defined. For variance, making use of a change of variable $y = (x-\mu)/\sigma$,

$$\begin{aligned} \text{var}(X) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy \\ &= \frac{\sigma^2}{\sqrt{2\pi}}\left[\left(-ye^{-y^2/2}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-y^2/2} dy\right] \\ &= \frac{\sigma^2}{\sqrt{2\pi}}\left(0 + \sqrt{2\pi}\right) = \sigma^2 \end{aligned}$$

(3) Normal random variable

If $X \sim \mathcal{N}(\mu, \sigma^2)$, and a linear transformation $Y = aX + b$ where $a \neq 0$ and b are scalars,

$$\boxed{\begin{cases} X \sim \mathcal{N}(\mu, \sigma^2) \\ Y = aX + b \end{cases} \implies Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)} \tag{16}$$

Corollary:

If $Y = \dfrac{X-\mu}{\sigma} = \dfrac{1}{\sigma}X - \dfrac{\mu}{\sigma}$, then $Y \sim \mathcal{N}(0, 1)$, called standard normal.

To see why, we'll look at cumulative density function and its use in deriving transforms of random variables first.

Example

Compute $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ for the following cases:

- $X = A\cos(\omega t + \theta)$, where $A \sim \mathcal{N}(\mu, \sigma^2)$
- $X = a\cos(\omega t + \Theta)$, where $\Theta \sim U(0, 2\pi)$
- $X = a\cos(\omega T + \theta)$, where $T \sim U(-\frac{\pi}{\omega}, \frac{\pi}{\omega})$

ANS: For $X = A\cos(\omega t + \theta)$:

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(A\cos(\omega t + \theta)) = \mu\cos(\omega t + \theta)\\ \mathrm{var}(X) &= \mathrm{var}(A\cos(\omega t + \theta)) = \sigma^2\cos^2(\omega t + \theta)\end{aligned}$$

Because $\mathrm{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$, we have

$$\mathbb{E}(X^2) = (\mu^2 + \sigma^2)\cos^2(\omega t + \theta)$$

Example

For $X = a\cos(\omega t + \Theta)$:

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(a\cos(\omega t + \Theta))\\ &= a\int_0^{2\pi}\cos(\omega t + \theta)\cdot\frac{1}{2\pi}d\theta\\ &= 0\end{aligned}$$

This is because we are integrating a full period of a cosine function, so the net area is zero.

Example

(cont.) For $X = a\cos(\omega t + \Theta)$:

$$\begin{aligned}\mathbb{E}(X^2) &= \mathbb{E}\left(a^2\cos^2(\omega t + \Theta)\right)\\ &= a^2\int_0^{2\pi}\cos^2(\omega t + \theta)\cdot\frac{1}{2\pi}d\theta\\ &= a^2\cdot\pi\cdot\frac{1}{2\pi}\\ &= \frac{a^2}{2}\end{aligned}$$

To compute the integral, you may want to use the relationship $\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$.

Example

For $X = a\cos(\omega T + \theta)$:

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(a\cos(\omega T + \theta))\\ &= a\int_{-\pi/\omega}^{\pi/\omega}\cos(\omega t + \theta)\cdot\frac{\omega}{2\pi}dt = 0\\ \mathbb{E}(X^2) &= \mathbb{E}\left(a^2\cos^2(\omega T + \theta)\right)\\ &= a^2\int_{-\pi/\omega}^{\pi/\omega}\cos^2(\omega t + \theta)\cdot\frac{\omega}{2\pi}dt\\ &= a^2\cdot\frac{1}{2}\left(\frac{\pi}{\omega} - \frac{-\pi}{\omega}\right)\cdot\frac{\omega}{2\pi} = \frac{a^2}{2}\end{aligned}$$

Again, we are integrating a full period of cosine (or cosine squared) functions.

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Definition

The cumulative density function (CDF) of a random variable X is denoted $F_X(x)$. This can be defined for both discrete and continuous r.v.:

$$F_X(x) = \mathbb{P}(X \leqslant x) = \begin{cases} \sum_{k \leqslant x} p_X(k) & X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(t)\,dt & X \text{ is continuous} \end{cases} \tag{17}$$

It is more common to use it for the continuous case.

Interpretation: CDF is the accumulated probability up to x.

Definition

Properties of CDF:

- CDF is nondecreasing.
- The maximum of the CDF is when $x = \infty$: $F_X(\infty) = 1$.
- The minimum of the CDF is when $x = -\infty$: $F_X(-\infty) = 0$.

Definition

We can also obtain PMF or PDF from CDF:

$$\begin{cases} p_X(k) = F_X(k) - F_X(k-1) & \text{discrete} \\ f_X(x) = \dfrac{d}{dx} F_X(x) & \text{continuous} \end{cases} \tag{18}$$

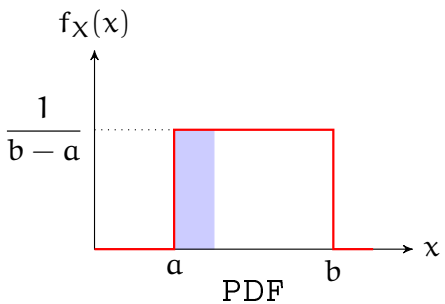
This is particularly useful when we define new r.v. through CDF, and then differentiate (for continuous r.v.) to obtain PDF.

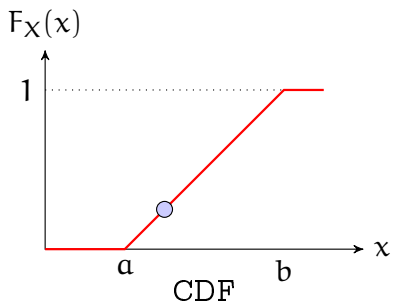
Example

For a uniform distribution $X \sim U(a, b)$, what is the CDF?

ANS: The CDF is given by

$$F_X(x) = \begin{cases} 0 & x \leqslant a \\ \int_{-\infty}^{x} f_X(x')\,dx' = \int_a^x \frac{1}{b-a}\,dx' = \frac{x-a}{b-a} & a < x \leqslant b \\ 1 & b < x \end{cases}$$





Example

Assume you take a quiz and the score is (discrete) uniform between 1 and 10. You are allowed to take the test three times and count only their maximum. What is the final PMF?

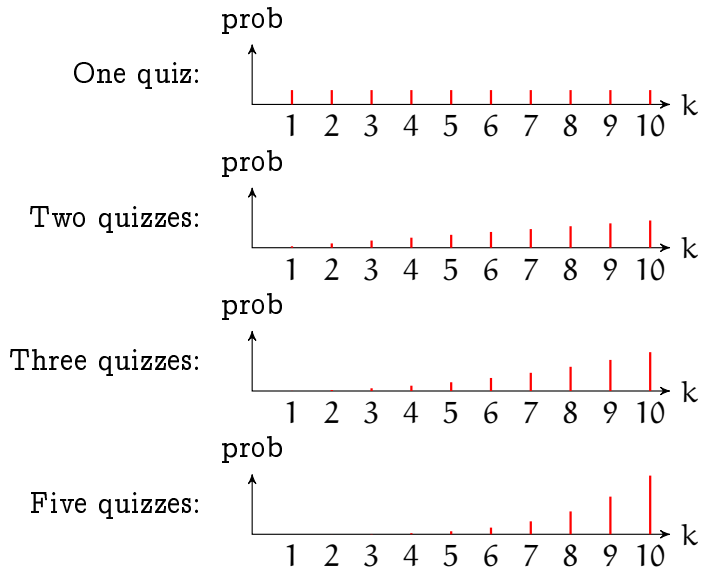
ANS: Let the quiz scores be X_1, X_2, X_3, so that $X = \max\{X_1, X_2, X_3\}$.

$$\begin{aligned} F_X(k) &= \mathbb{P}(X \leqslant k) = \mathbb{P}(X_1 \leqslant k, X_2 \leqslant k, X_3 \leqslant k) \\ &= \mathbb{P}(X_1 \leqslant k)\,\mathbb{P}(X_2 \leqslant k)\,\mathbb{P}(X_3 \leqslant k) = \left(\frac{k}{10}\right)^3 \end{aligned}$$

since the quiz score for each test is independent of each other.

$$\begin{aligned} p_X(k) &= F_X(k) - F_X(k-1) \\ &= \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3 \qquad k = 1, \ldots, 10 \end{aligned}$$

Example



Geometric CDF

Remember $X \sim \text{Geo}(p)$ has PMF

$$p_X(k) = (1-p)^{k-1}p \qquad k = 1, 2, \ldots$$

Its CDF is

$$\boxed{F_X(n) = \sum_{k=1}^{n} (1-p)^{k-1}p = \frac{1-(1-p)^n}{1-(1-p)}p = 1-(1-p)^n} \tag{19}$$

for $n = 1, 2, \ldots$

Exponential CDF

$X \sim \text{Exp}(\lambda)$ has PDF

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \geqslant 0$$

Its CDF is

$$\boxed{F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}} \tag{20}$$

for $x > 0$.

Geometric and exponential CDF

Geometric and exponential CDFs are closely related. By defining $\delta = -\ln(1-p)/\lambda$, we have

$$e^{-\lambda\delta} = 1 - p$$

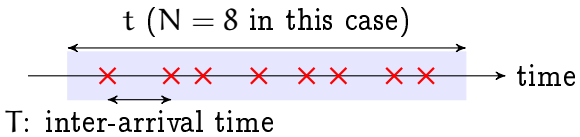
and whenever $x = n\delta$,

$$\boxed{F_{exp}(n\delta) = 1 - e^{-\lambda n\delta} = 1 - (1-p)^n = F_{geo}(n)} \tag{21}$$

Interpretation: Suppose we toss very quickly (every δ seconds, where $\delta \ll 1$) a biased coin with a very small probability of heads (equal to $p = 1 - e^{-\lambda\delta}$). The first time to obtain head, which is a geometric r.v. with parameter p, is a close approximation to an exponential r.v. with parameter λ.

Connecting exponential and Poisson random variables

Exponential random variable often has to do with *time*. It is also intimately related to the Poisson process. They are the foundations of a subject area known as queuing theory.



We start with Poisson, where people arrive at a rate of λ per unit time. Thus, within a time t, the number of arrival is $N \sim \text{Poisson}(\lambda t)$, i.e.,

$$\mathbb{P}(N = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

We are interested to find the *inter-arrival time* between consecutive people. Let this be T, which is a random variable.

Connecting exponential and Poisson random variables

We aim to find the CDF of T. We first note that

$$\begin{aligned}\mathbb{P}(T > t) &= \mathbb{P}(\text{inter-arrival time} > t)\\ &= \mathbb{P}(\text{no arrival within } t)\\ &= \mathbb{P}(N = 0 \text{ within } t)\end{aligned}$$

The last probability is Poisson, with $\mathbb{P}(N = 0) = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}$.

The CDF $F_T(t)$ computes the total probability that T is smaller than t, so

$$F_T(t) = 1 - \mathbb{P}(T > t) = 1 - e^{-\lambda t}$$

Therefore, $T \sim \text{Exp}(t)$.

Normal CDF

For $X \sim \mathcal{N}(0,1)$, the CDF is

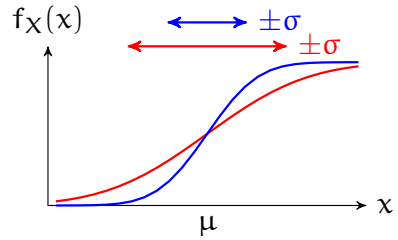
$$\boxed{F_X(x) = \Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt} \tag{22}$$

We cannot analytically integrate the function, but we can compute it as a function $\Phi(x)$ beforehand.

Then, for $X \sim \mathcal{N}(\mu, \sigma^2)$, we can "standardize" by defining $Y = \frac{X-\mu}{\sigma}$ such that $\mathbb{E}(Y) = 0$ and $\text{var}(Y) = 1$, and

$$\boxed{\mathbb{P}(X \leqslant x) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{x-\mu}{\sigma}\right) = \mathbb{P}\left(Y \leqslant \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)} \tag{23}$$

(3) Normal CDF



Example

A binary signal $\{+1, -1\}$ is transmitted through a channel corrupted with additive Gaussian noise $N \sim \mathcal{N}(0, \sigma^2)$. The receiver decides the signal is $+1$ if the received signal is $\geqslant 0$, and -1 if the received signal is < 0. What is the probability of error?

ANS: When the signal is -1, error occurs if $N \geqslant 1$.

$$\mathbb{P}(N \geqslant 1) = 1 - \mathbb{P}(N < 1) = 1 - \Phi\left(\frac{1}{\sigma}\right)$$

By symmetry, we have the same probability of error when signal is $+1$, and error occurs if $N < -1$.

As example, when $\sigma = 1$, $\Phi(1) \approx 0.84$, and probability of error is ≈ 0.16.

Median, Mode, Mean

The CDF also gives us a direct way to find some useful statistics on the distribution.

Median: The value c that separates the "higher half" and "lower half" of the distribution:

$$\boxed{\int_{-\infty}^{c} f_X(x)\, dx = \int_{c}^{\infty} f_X(x)\, dx} \tag{24}$$

Using CDF, that means:

$$\boxed{F_X(c) = \frac{1}{2}} \tag{25}$$

Example

What is the median of (i) $X \sim U(a, b)$; (ii) $X \sim \text{Exp}(\lambda)$?

ANS: (i) For a uniform distribution, $F_X(x) = \frac{x-a}{b-a}$ for $a < x \leqslant b$. Therefore,

$$F_X(c) = \frac{c-a}{b-a} = \frac{1}{2} \qquad \Rightarrow \qquad c = \frac{a+b}{2}$$

(ii) For an exponential distribution, $F_X(x) = 1 - e^{-\lambda x}$ for $x > 0$. Therefore,

$$F_X(c) = 1 - e^{-\lambda c} = \frac{1}{2} \qquad \Rightarrow \qquad c = \frac{1}{\lambda} \log 2$$

Median, Mode, Mean

Mode: The value c associated with the peak of the PDF:

$$\boxed{c = \arg\max_x f_X(x)} \tag{26}$$

(This notation means we want to find the maximum of $f_X(x)$, but we are mostly interested in the argument x that leads to this maximum, and assign c to this value.) Using CDF, that means:

$$\boxed{c = \arg\max_x \frac{d}{dx} F_X(x)} \tag{27}$$

Example

Find the mode of $F_X(x) = \begin{cases} 0 & x \leqslant 0 \\ 3x^2 - 2x^3 & 0 < x \leqslant 1 \\ 1 & x > 1 \end{cases}$.

ANS: For $0 < x \leqslant 1$, we have $f_X(x) = \frac{d}{dx} F_X(x) = 6x - 6x^2$. To find its maximum, we take the derivative and set to zero,

$$\frac{d}{dx}(6x - 6x^2) = 6 - 12x = 0 \qquad \Rightarrow \qquad x = \frac{1}{2}$$

To confirm that it is a maximum, we can take the second derivative and note that it is negative:

$$\frac{d^2}{dx^2}(6x - 6x^2) = -12$$

So, $c = \frac{1}{2}$ is the mode.

Median, Mode, Mean

Mean: We can also find the mean directly from the CDF:

$$\boxed{\mathbb{E}(X) = \int_0^\infty \big(1 - F_X(t)\big)\,dt - \int_{-\infty}^0 F_X(t)\,dt} \tag{28}$$

The proof is a little bit more involved and we do it in three steps. The discussion (and pictures) follow Section 4.4 of the reference text.

Median, Mode, Mean

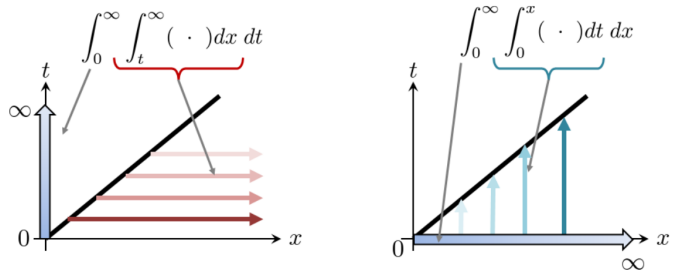
(I) Let $X \geqslant 0$. We want to show $\mathbb{E}(X) = \int_0^\infty \big(1 - F_X(t)\big)\,dt$.

We start with the right hand side:

$$\int_0^\infty \big(1 - F_X(t)\big)\,dt = \int_0^\infty \big(1 - \mathbb{P}(X \leqslant t)\big)\,dt = \int_0^\infty \big(\mathbb{P}(X > t)\big)\,dt$$
$$= \int_0^\infty \left(\int_t^\infty f_X(x)\,dx\right) dt$$

The trick is to change the integration order.

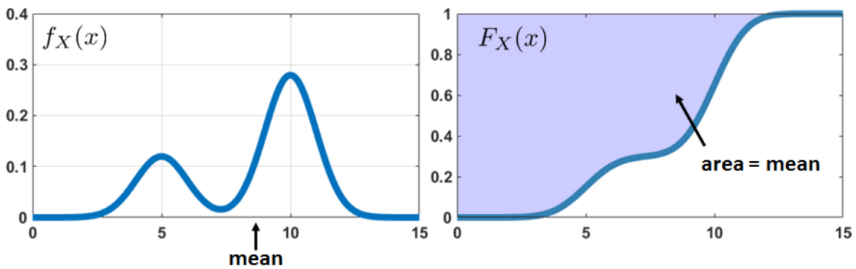
Median, Mode, Mean



$$\int_0^\infty \int_t^\infty f_X(x)\,dx dt = \int_0^\infty \int_0^x f_X(x)\,dt dx$$
$$= \int_0^\infty f_X(x) \left(\int_0^x dt\right) dx$$
$$= \int_0^\infty x f_X(x)\,dx$$
$$= \mathbb{E}(X)$$

Median, Mode, Mean

Interpretation of $\mathbb{E}(X) = \int_0^\infty \big(1 - F_X(t)\big)\,dt$:



Median, Mode, Mean

(II) Let $X < 0$. We want to show $\mathbb{E}(X) = -\int_{-\infty}^0 F_X(t)\,dt$.

We again start with the right hand side, and change the integration order:

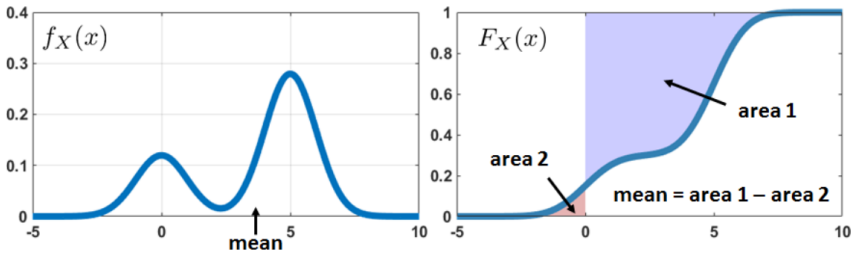
$$\int_{-\infty}^0 F_X(t)\,dt = \int_{-\infty}^0 \int_{-\infty}^t f_X(x)\,dx dt$$
$$= \int_{-\infty}^0 \int_x^0 f_X(x)\,dt dx$$
$$= \int_{-\infty}^0 -x f_X(x)\,dx$$
$$= -\mathbb{E}(X)$$

Median, Mode, Mean

(III) For a general X, we let $X = X^+ + X^-$, where X^+ is the positive part and X^- is the negative part. Then,

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(X^+) + \mathbb{E}(X^-) \\ &= \int_0^\infty (1 - F_X(t))\,dt - \int_{-\infty}^0 F_X(t)\,dt\end{aligned}$$

Interpretation:



1. Expectation, mean, variance
2. Common continuous random variables
3. Cumulative density function
4. Derived distributions
5. Joint PDF, conditioning, independence
6. Continuous Bayes' rule

Procedure

Given: PDF of X, and $Y = g(X)$, find the PDF of Y.

A two-step procedure:

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(g(X) \leqslant y) = \int_{\{x \mid g(x) \leqslant y\}} f_X(x)\,dx \tag{29}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) \tag{30}$$

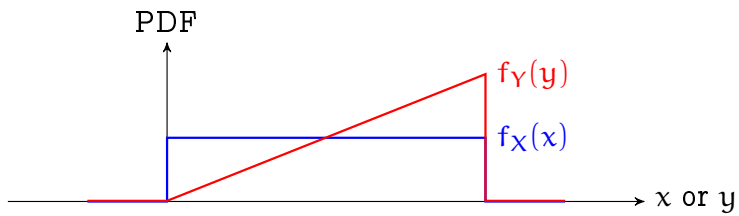
Example

$X \sim U(0,1)$ and $Y = \sqrt{X}$. Find PDF of Y.

ANS: Follow the procedure to find $F_Y(y)$ and $f_Y(y)$.

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}\left(\sqrt{X} \leqslant y\right) = \mathbb{P}(X \leqslant y^2) = y^2$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(y^2) = 2y \qquad 0 \leqslant y \leqslant 1$$



Example

$X \sim U(5, 10)$ and $Y = \dfrac{1}{X}$. Find PDF of Y.

ANS: Procedure is similar, but we need $F_X(x)$ first.

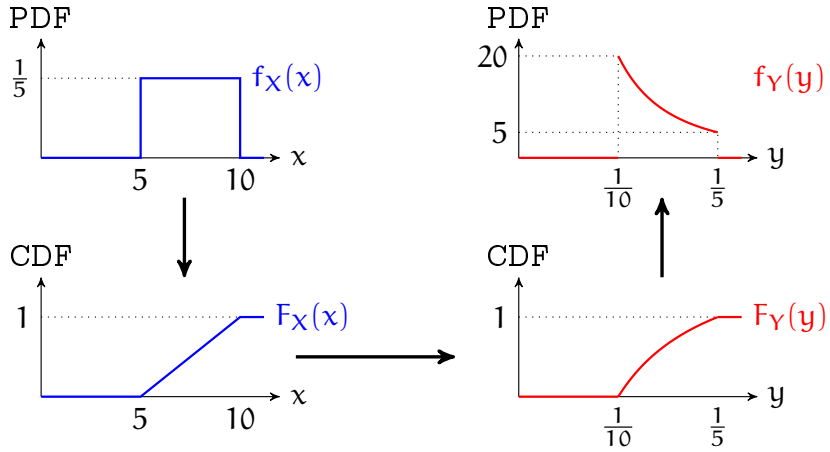
$$F_X(x) = \begin{cases} 0 & x \leqslant 5 \\ \frac{x-5}{5} & 5 < x \leqslant 10 \\ 1 & 10 < x \end{cases}$$

Then, $F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}\left(\dfrac{1}{X} \leqslant y\right) = \mathbb{P}\left(\dfrac{1}{y} \leqslant X\right) = 1 - F_X\left(\dfrac{1}{y}\right)$.

$$F_Y(y) = \begin{cases} 0 & y \leqslant \frac{1}{10} \\ 1 - (\frac{1}{y} - 5)/5 = 2 - \frac{1}{5y} & \frac{1}{10} < y \leqslant \frac{1}{5} \\ 1 & \frac{1}{5} < y \end{cases}$$

Example

Differentiating, $f_Y(y) = \begin{cases} 0 & y \leqslant \frac{1}{10} \quad \text{or} \quad \frac{1}{5} < y \\ \frac{1}{5y^2} & \frac{1}{10} < y \leqslant \frac{1}{5} \end{cases}$



Example

X has known PDF, and $Y = X^2$. Find PDF of Y.

ANS: For $y \geqslant 0$,

$$\begin{aligned} F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(X^2 \leqslant y) &= \mathbb{P}(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

Differentiating,

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) \qquad y \geqslant 0$$

Special case: Y is a linear function of X

Given: PDF of X, and $Y = aX + b$, find the PDF of Y.

Case 1: $a > 0$

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(aX + b \leqslant y) = \mathbb{P}\left(X \leqslant \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

Case 2: $a < 0$

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(aX + b \leqslant y) = \mathbb{P}\left(X \geqslant \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

Special case: Y is a linear function of X

Given: PDF of X, and $Y = aX + b$,

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)} \tag{31}$$

Example: $X \sim \mathcal{N}(\mu, \sigma^2)$

$$\begin{aligned} f_Y(y) &= \frac{1}{|a|}\frac{1}{\sqrt{2\pi}\sigma} e^{-(\frac{y-b}{a}-\mu)^2/2\sigma^2} \\ &= \frac{1}{\sqrt{2\pi}|a|\sigma} e^{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}} \end{aligned}$$

i.e., $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

1. Expectation, mean, variance
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Definition

Consider two continuous r.v. X and Y, with joint PDF

$$\boxed{\mathbb{P}((X,Y) \in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy} \tag{32}$$

for every subset B of the 2D plane.
When $B = \{(x,y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$,

$$\boxed{\mathbb{P}(a \leqslant x \leqslant b, c \leqslant y \leqslant d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy} \tag{33}$$

Definition

We can calculate the marginal PDFs of X and Y using

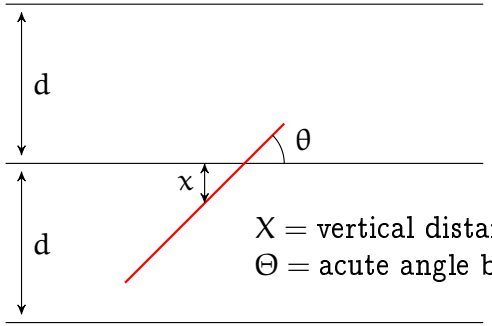
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \tag{34}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \tag{35}$$

and, for normalization, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) = 1$

Example

Buffon's Needle. We have many parallel lines at distances d apart. Assume we throw a needle of length ℓ, where $\ell < d$, on the surface at random. What is the probability that the needle will intersect one of the lines?



$X =$ vertical distance from mid needle to a line
$\Theta =$ acute angle between needle and line

$\vdots$

Example

At random: *assume a uniform joint PDF over the rectangular set*

$$S = \left\{ (x, \theta) \,\middle|\, 0 \leqslant x \leqslant \frac{d}{2}, 0 \leqslant \theta \leqslant \frac{\pi}{2} \right\}$$

This rectangle has an area $\pi d/4$, and X and Θ are independent, so

$$f_{X,\Theta}(x, \theta) = \begin{cases} \frac{4}{\pi d} & (x, \theta) \in S \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\frac{\ell}{2} \sin \Theta$ is the *vertical* distance between the middle of the needle to the end points of the needle; the needle will intersect one of the lines if and only if

$$X \leqslant \frac{\ell}{2} \sin \Theta$$

Example

$$\begin{aligned} \mathbb{P}\left(X \leqslant \frac{\ell}{2} \sin \Theta\right) &= \int_0^{\pi/2} \int_0^{(\ell/2) \sin \theta} \frac{4}{\pi d} dx d\theta \\ &= \frac{4}{\pi d} \int_0^{\pi/2} \frac{\ell}{2} \sin \theta d\theta \\ &= \frac{2\ell}{\pi d} (-\cos \theta) \Big|_0^{\pi/2} \\ &= \frac{2\ell}{\pi d} \end{aligned}$$

This can be used as a Monte Carlo method to find π: if we drop n needles and h of them cross lines, then

$$\pi \approx \frac{2\ell}{d} \cdot \frac{n}{h}$$

Conditioning on an event

We can have conditional PDF of a continuous r.v. X given event A

$$\boxed{\mathbb{P}(X \in B \,|\, A) = \int_B f_{X|A}(x)\, dx} \tag{36}$$

with $\int_{-\infty}^{\infty} f_{X|A}(x)\, dx = 1$. Specifically, if the event is of the form $\{X \in A\}$ with $\mathbb{P}(X \in A) > 0$,

$$\mathbb{P}(X \in B \,|\, X \in A) = \frac{\mathbb{P}(X \in B, X \in A)}{\mathbb{P}(X \in A)} = \frac{\int_{A \cap B} f_X(x)\, dx}{\mathbb{P}(X \in A)}$$

$$\boxed{f_{X|\{X \in A\}}(x) = \begin{cases} \frac{f_X(x)}{\mathbb{P}(X \in A)} & x \in A \\ 0 & \text{otherwise.} \end{cases}} \tag{37}$$

Interpretation: Integrate all cases of $X = x$ AND belonging to A, then normalize with $\mathbb{P}(X \in A)$.

Example

A certain device has a lifetime following an exponential r.v. with parameter λ. Suppose now we know it has remained in operation for time t; and let X be the *additional* time until it fails. What is the conditional PDF of X, given it has already lasted for time t?

ANS: Define event $A = \{T > t\}$. Let us look at $\mathbb{P}(X > x \mid A)$.

$$\begin{aligned}\mathbb{P}(X > x \mid A) = \mathbb{P}(T > t + x \mid T > t) &= \frac{\mathbb{P}(T > t + x \text{ and } T > t)}{\mathbb{P}(T > t)} \\ &= \frac{\mathbb{P}(T > t + x)}{\mathbb{P}(T > t)} \\ &= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}\end{aligned}$$

So $\mathbb{P}(X < x \mid A) = 1 - e^{-\lambda x}$, which means $X \sim \text{Exp}(\lambda)$.

Interpretation: Exponential r.v. is memoryless.

Conditioning on a r.v.

The conditional PDF of a r.v. X given knowledge of another r.v. Y

$$\boxed{f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}} \tag{38}$$

similarly, $\int_{-\infty}^{\infty} f_{X \mid Y}(x \mid y)\, dx = 1$.

Example

A dart lands uniformly random on a circular target of radius r. Let x and y be the Cartesian coordinates. Find the marginal PDF $f_Y(y)$ and conditional PDF $f_{X \mid Y}(x \mid y)$.

ANS: The joint PDF is $f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi r^2} & x^2 + y^2 \leqslant r^2 \\ 0 & \text{otherwise.} \end{cases}$

The marginal PDF is

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx &= \frac{1}{\pi r^2} \int_{x^2+y^2 \leqslant r^2} dx \\ &= \frac{1}{\pi r^2} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx \\ &= \frac{2}{\pi r^2} \sqrt{r^2 - y^2} \qquad \text{if } |y| \leqslant r \end{aligned}$$

This is NOT uniform!

Example

The conditional PDF is

$$\begin{aligned} f_{X \mid Y}(x \mid y) &= \frac{f_{X,Y}(x, y)}{f_Y(y)} \\ &= \frac{\frac{1}{\pi r^2}}{\frac{2}{\pi r^2}\sqrt{r^2 - y^2}} \\ &= \frac{1}{2\sqrt{r^2 - y^2}} \qquad \text{if } x^2 + y^2 \leqslant r^2 \end{aligned}$$

Given a fixed y, this is uniform!

Conditional expectation

Let X and Y be two continuous r.v., and A an event. We can compute the conditional expectations:

$$\mathbb{E}(X \mid A) = \int_{-\infty}^{\infty} x f_{X \mid A}(x)\, dx \tag{39}$$

$$\mathbb{E}(g(X) \mid A) = \int_{-\infty}^{\infty} g(x) f_{X \mid A}(x)\, dx \tag{40}$$

$$\mathbb{E}(X \mid Y) = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y)\, dx \tag{41}$$

$$\mathbb{E}(g(X) \mid Y) = \int_{-\infty}^{\infty} g(x) f_{X \mid Y}(x \mid y)\, dx \tag{42}$$

Conditional expectation

Assume $A_1, \ldots, A_n$ are disjoint events partitioning the sample space, with $\mathbb{P}(A_i) > 0$ for all i.

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{P}(A_i)\, \mathbb{E}(X \mid A_i) \tag{43}$$

unconditional average = averaging the conditional averages

Also when $A_i = \{Y = y\}$,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} \mathbb{E}(X \mid Y = y)\, f_Y(y)\, dy \tag{44}$$

Conditional expectation

Extensions to functions of several r.v.:

$$\mathbb{E}(g(X,Y) \mid Y) = \int_{-\infty}^{\infty} g(x,y) f_{X \mid Y}(x \mid y)\, dx \tag{45}$$

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{\infty} \mathbb{E}(g(X,Y) \mid Y)\, f_Y(y)\, dy \tag{46}$$

Independence

Two r.v. X and Y are independent of each other,

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \tag{47}$$

Since $f_{X,Y}(x,y) = f_{X \mid Y}(x \mid y)\, f_Y(y)$, for all y with $f_Y(y) > 0$,

$$f_{X \mid Y}(x \mid y) = f_X(x) \tag{48}$$

Can also interchange X and Y. Furthermore,

$$\mathbb{E}(XY) = \mathbb{E}(X)\, \mathbb{E}(Y) \tag{49}$$

$$\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X))\, \mathbb{E}(h(Y)) \tag{50}$$

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) \tag{51}$$

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Definition

Unobserved phenomenon: r.v. X with $f_X(x)$
Measurement: Y, related to X through $f_{Y|X}(y|x)$
From the observation $Y = y$, what do we know about X?
$\Rightarrow$ we want to know $f_{X|Y}(x|y)$

$$\boxed{f_{X|Y}(x|y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}} \tag{52}$$

This is the continuous form of Bayes' rule. Alternatively,

$$\boxed{f_{X|Y}(x|y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t)\, f_{Y|X}(y|t)\, dt}} \tag{53}$$

Example

Suppose we have electronic devices with exponential lifetime $Y \sim \text{Exp}(\lambda)$, but λ *itself is a random variable*, modeled by uniform distribution $\Lambda \sim U(1,3)$. Given we observe the lifetime of a particular device to be $Y = y$, what do we know about λ?

ANS: We want to find $f_{\Lambda|Y}(\lambda|y)$ through Bayes' rule.

$$f_\Lambda(\lambda) = \tfrac{1}{3-1} = \tfrac{1}{2} \qquad 1 \leqslant \lambda \leqslant 3$$
$$f_{Y|\Lambda}(y|\lambda) = \lambda e^{-\lambda y}$$
$$\int_{-\infty}^{\infty} f_\Lambda(t)\, f_{Y|\Lambda}(y|t)\, dt = \int_1^3 \tfrac{1}{2} t e^{-ty}\, dt$$

Example

Integration by parts:

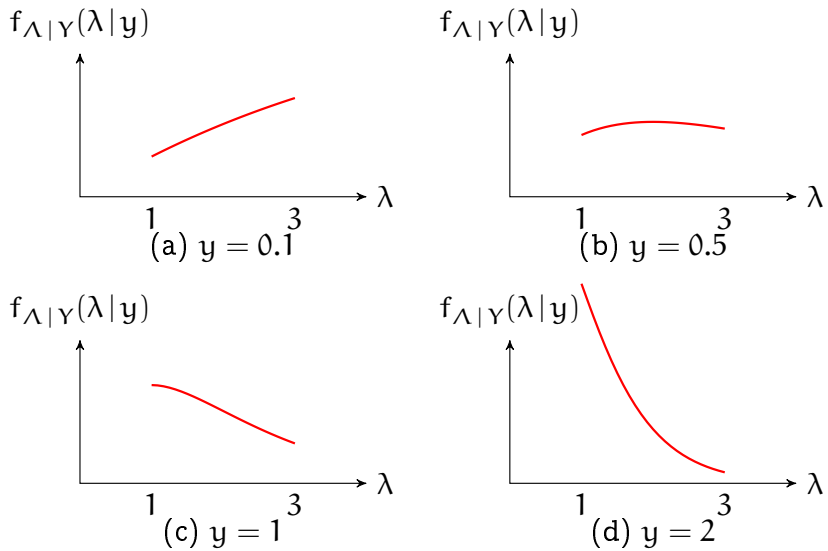
$$\int_a^b v \frac{du}{dt} dt = uv\Big|_a^b - \int_a^b u \frac{dv}{dt} dt$$

Let $v = t$ and $u = -\frac{1}{y} e^{-yt}$, $a = 1$ and $b = 3$,

$$\begin{aligned}
\tfrac{1}{2}\int_1^3 t e^{-yt} dt &= -\tfrac{t}{2y} e^{-yt}\Big|_1^3 - \int_1^3 -\tfrac{1}{2y} e^{-yt} dt \\
&= \tfrac{1}{2}\left(-\tfrac{3}{y} e^{-3y} + \tfrac{1}{y} e^{-y}\right) - \tfrac{1}{2y^2} e^{-yt}\Big|_1^3 \\
&= \tfrac{1}{2y}\left(e^{-y} - 3e^{-3y}\right) + \tfrac{1}{2y^2}\left(e^{-y} - e^{-3y}\right)
\end{aligned}$$

$$f_{\Lambda|Y}(\lambda|y) = \frac{\lambda e^{-\lambda y}}{\frac{1}{y}\left(e^{-y} - 3e^{-3y}\right) + \frac{1}{y^2}\left(e^{-y} - e^{-3y}\right)} \qquad 1 \leqslant \lambda \leqslant 3$$

Example



Special case 1: Unobserved phenomenon is discrete

Unobserved phenomenon: event A with $\mathbb{P}(A)$
Measurement: Y, related to A through $f_{Y|A}(y)$
From the observation $Y = y$, what do we know about A?
$\Rightarrow$ we want to know $\mathbb{P}(A \,|\, Y = y)$

$$\boxed{\mathbb{P}(A \,|\, Y = y) = \frac{\mathbb{P}(A)\, f_{Y|A}(y)}{f_Y(y)}} \tag{54}$$

Special case 1: Unobserved phenomenon is discrete

Case 1a: A is "binary", so $f_Y(y) = \mathbb{P}(A)\, f_{Y|A}(y) + \mathbb{P}(A^c)\, f_{Y|A^c}(y)$

$$\boxed{\mathbb{P}(A \,|\, Y = y) = \frac{\mathbb{P}(A)\, f_{Y|A}(y)}{\mathbb{P}(A)\, f_{Y|A}(y) + \mathbb{P}(A^c)\, f_{Y|A^c}(y)}} \tag{55}$$

Case 1b: A is of the form $\{N = n\}$, where $n = 1, \ldots, K$, with $p_N(n)$

$$\boxed{\mathbb{P}(N = n \,|\, Y = y) = \frac{p_N(n)\, f_{Y|N}(y|n)}{\sum_{i=1}^{K} p_N(i)\, f_{Y|N}(y|i)}} \tag{56}$$

Example

A binary signal S with $\mathbb{P}(S = 1) = p$ and $\mathbb{P}(S = -1) = 1 - p$ is transmitted, which is corrupted with independent Gaussian noise $N \sim \mathcal{N}(0, 1)$, and the received signal is $Y = N + S$. What is the probability of $S = 1$ given an observation $Y = y$?

ANS: Let $A = \{S = 1\}$.

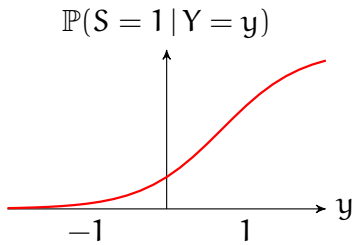
$$f_{Y|A}(y) \sim \mathcal{N}(1, 1)$$
$$f_{Y|A^c}(y) \sim \mathcal{N}(-1, 1)$$

$$\mathbb{P}(A \,|\, Y = y) = \frac{p \cdot \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2}}{p \cdot \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2} + (1-p) \cdot \frac{1}{\sqrt{2\pi}} e^{-(y+1)^2/2}}$$
$$= \frac{p e^{y}}{p e^{y} + (1-p) e^{-y}}$$

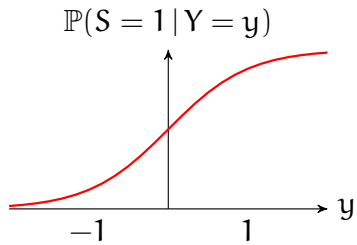
Example

$$\mathbb{P}(S = 1 \mid Y = y) = \frac{pe^{y}}{pe^{y} + (1-p)e^{-y}}$$

Let us consider specific values of p:



(a) $p = 0.2$

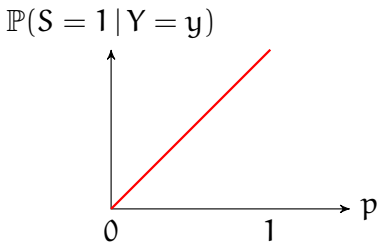


(b) $p = 0.5$

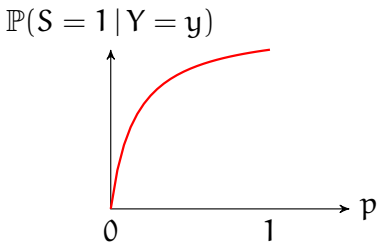
Example

$$\mathbb{P}(S = 1 \mid Y = y) = \frac{pe^{y}}{pe^{y} + (1-p)e^{-y}}$$

Let us consider specific values of y:



(a) $y = 0$



(b) $y = 1$

Conclusions

We developed the basic toolset for continuous random variables, and looked at some common distributions.