

Fundamentals of Probability

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<https://www.eee.hku.hk/~elec2844>

Calibration questions

[Birthday Party problem]

Question: A year has 365 days. In a room with N people, what is the smallest value of N such that the probability of two or more people having birthday on the same day exceeds $\frac{1}{2}$?

ANS: 23

https://en.wikipedia.org/wiki/Birthday_problem

Meaning of probability

Does “probability” have different meaning depending on the question?

① I roll a die and there is $\frac{1}{6}$ chance I get .

② There is $\frac{1}{6}$ chance Hang Seng Index will go down tomorrow.

Calibration questions

[Deck shuffling problem]

Question: You are given a deck of cards organized sequentially. How many shuffles does it take to result in a fully random deck?

ANS: 7

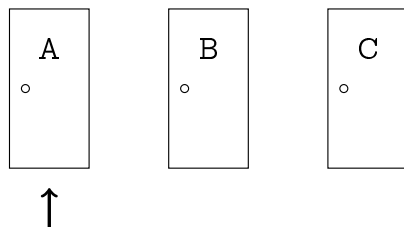
<https://www.math.hmc.edu/funfacts/ffiles/20002.4-6.shtml>

https://www.dartmouth.edu/~chance/teaching_aids/Mann.pdf

Calibration questions

[Monty Hall problem]

Question: There is one prize hidden behind one of three doors with equal likelihood. You choose one door. The host opens one of the remaining two doors and reveals that there is no prize behind that door. You are now offered a chance to switch. Will you?



ANS: Yes!

https://en.wikipedia.org/wiki/Monty_Hall_problem

Calibration questions

Problems involving probability can be

- counterintuitive
- ill-defined
- complicated

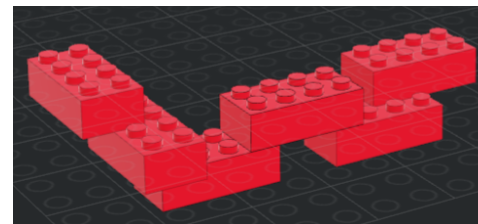
but also

- important
- powerful

especially in this era of big data and machine learning

Random bricks

You and I were both given six 2×4 red bricks. I made the following construction:



What is the probability that you and I made the same construction? (Not counting rotation or symmetry)

With 6 bricks, you can get 915,103,765 combinations! So the probability is $1/915103765$. (This particular construction is #658,916,208 according to Lego. See also <http://web.math.ku.dk/~eilers/lego.html>)

1 Sets and probability models

2 Conditional probability

3 Total probability and Bayes' rule

4 Independence

5 Counting

Mathematical foundation

We have some intuitive understanding of “chance”, but we need a mathematical formalism so we can agree on more precise meaning.

- I am “110% sure” you want to know this!
- In multiple choice questions, can you be 100% sure about your answer?

We also need mathematical tools to help us manipulate expressions and do various derivations. *Set theory* turns out to be the appropriate mathematics for the task.

Sets — relations and operations

- Set S is a subset of T :

$$S \subset T \quad \text{or, if they can be equal,} \quad S \subseteq T \quad (5)$$

- Null set $\emptyset$ and universal set Ω
- Set complement:

$$S^c = \{x \in \Omega \mid x \notin S\} \quad (6)$$

- Set union:

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\} \quad (7)$$

$$\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup \dots \quad (8)$$

- Set intersection:

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\} \quad (9)$$

$$\bigcap_{n=1}^{\infty} S_n = S_1 \cap S_2 \cap \dots \quad (10)$$

Sets — definitions

- Set S with elements x_i :

$$S = \{x_1, x_2, \dots, x_n\} \quad \text{and also} \quad x_i \in S \text{ for } 1 \leq i \leq n \quad (1)$$

- A set can be finite, e.g.,

$$S = \{H, T\} \quad \text{or} \quad S = \{1, 2, 3, 4, 5, 6\} \quad (2)$$

- A set can be countably infinite, e.g.,

$$S = \{2, 4, 6, \dots\} \quad \text{set of all positive even integers} \quad (3)$$

- A set can be uncountable, e.g.,

$$S = \{x \mid 0 \leq x \leq 1\} \quad (4)$$

The symbol “ $\mid$ ” means “such that”

Sets — algebra

- Some equations:

$$S \cup T = T \cup S \quad (11)$$

$$S \cup (T \cap U) = (S \cup T) \cap (S \cup U) \quad (12)$$

$$S \cap (T \cup U) = (S \cap T) \cup (S \cap U) \quad (13)$$

$$S \cup (T \cap U) = (S \cup T) \cap (S \cup U) \quad (14)$$

$$(S^c)^c = S \quad (15)$$

$$S \cap S^c = \emptyset \quad (16)$$

$$S \cup \Omega = \Omega \quad (17)$$

$$S \cap \Omega = S \quad (18)$$

- De Morgan's theorems:

$$\left(\bigcup_n S_n \right)^c = \bigcap_n S_n^c \quad \left(\bigcap_n S_n \right)^c = \bigcup_n S_n^c \quad (19)$$

Probabilistic models

Fundamental math structure is a set:

- **Sample space** Ω : set of all possible **outcomes** or **events**

Law of Probability:

- Let A = a set of possible outcomes
- Map A to a nonnegative number $\mathbb{P}(A)$ called the **probability of A**
- This number reflects our belief of the “likelihood” of occurrence of such outcomes in an **experiment**

Properties of probability laws

Let A , B , and C be events (not necessarily disjoint):

$$A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B) \quad (23)$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad (24)$$

$$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B) \quad (25)$$

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(A^c \cap B) + \mathbb{P}(A^c \cap B^c \cap C) \quad (26)$$

Probability axioms

Three axioms: (Kolmogorov axioms)

- 1 **Nonnegativity**: for every event A

$$\mathbb{P}(A) \geq 0 \quad (20)$$

- 2 **Additivity**: if A and B are disjoint events

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \quad (21)$$

- 3 **Normalization**:

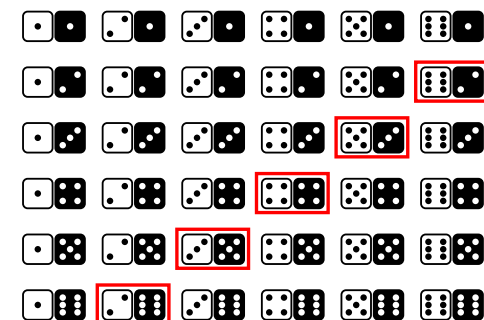
$$\mathbb{P}(\Omega) = 1 \quad (22)$$

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(\Omega \cup \emptyset) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset) = 1 + \mathbb{P}(\emptyset) \Rightarrow \mathbb{P}(\emptyset) = 0$$

Discrete model

Discrete model: “Count” the probability of individual outcome, and sum them up for the probability of a collection of outcomes (= event)

Example: Roll two dice. What is the probability that the sum is 8?

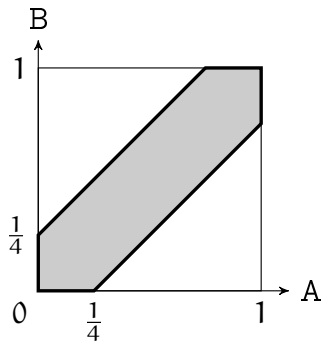


$$\text{Ans: } \mathbb{P}(\text{sum} = 8) = \frac{5}{36}$$

Continuous model

Continuous model: “Measure” the probability of individual interval, and integrate them for the probability of an event

Example: A and B agreed to meet at 3pm, but each will be late somewhere between 0 and 1 hour. *All pairs of delays are equally likely*. Whoever is first to arrive will wait for 15 min, and will leave if the other has not arrived. What is the probability that they meet?



$$\text{Ans: } \mathbb{P}(\text{A and B will meet}) = \frac{7}{16}$$

Conditional probability

Conditional probability gives us a formal way to assess the idea of “information”, and how that influences our computation of probability for certain events. Intuitively,

- Some information can be useless
- Some information may tell us everything
- Most interesting cases are in between, i.e., the new knowledge can be regarded as **partial information**, which adjusts our probability calculations

- 1 Sets and probability models
- 2 **Conditional probability**
- 3 Total probability and Bayes' rule
- 4 Independence
- 5 Counting

Definition

Consider a case where we have partial information. e.g. What is the probability of the first die being 3 if the sum of two dice is 8?

$$\mathbb{P}(A|B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B} \quad (27)$$

Thus we define conditional probability:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad (28)$$

To comply with our mathematical formalism, we can show that $\mathbb{P}(A|B)$ defined above follows the probability law.

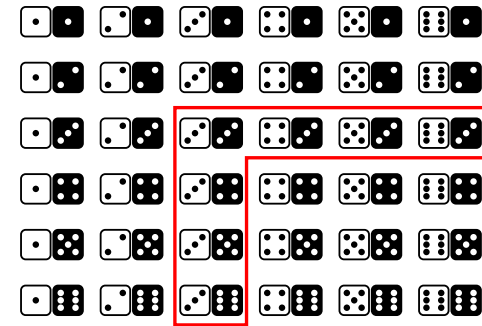
Examples

- ① Roll two dice, with outcomes X and Y . Let events

$$A = \{\max(X, Y) = 5\} \quad B = \{\min(X, Y) = 3\}.$$

Find $\mathbb{P}(A|B)$.

- ② Two students enter a competition. Probability that X wins a prize is $2/3$. Probability that Y wins a prize is $1/2$. Probability that at least one of them wins a prize is $3/4$. Now we know exactly one of them indeed wins a prize. What is the probability that it is X (or Y)?



$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{2/36}{7/36} = \frac{2}{7}$$

Examples

We use uppercase letters for winning, lowercase for losing. There are four outcomes: XY , Xy , xY , xy . We know:

$$\mathbb{P}(XY) + \mathbb{P}(Xy) = \frac{2}{3} \quad (X \text{ wins})$$

$$\mathbb{P}(XY) + \mathbb{P}(xY) = \frac{1}{2} \quad (Y \text{ wins})$$

$$\mathbb{P}(XY) + \mathbb{P}(Xy) + \mathbb{P}(xY) = \frac{3}{4} \quad (\text{at least one wins})$$

$$\mathbb{P}(XY) + \mathbb{P}(Xy) + \mathbb{P}(xY) + \mathbb{P}(xy) = 1 \quad (\text{all cases})$$

So we know $\mathbb{P}(xy) = \frac{1}{4}$, $\mathbb{P}(xY) = \frac{1}{12}$, $\mathbb{P}(Xy) = \frac{1}{4}$, $\mathbb{P}(XY) = \frac{5}{12}$.

$$\mathbb{P}(Xy | \{xY, Xy\}) = \frac{\frac{1}{4}}{\frac{1}{12} + \frac{1}{4}} = \frac{3}{4}$$

$$\mathbb{P}(xY | \{xY, Xy\}) = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{4}} = \frac{1}{4}$$

Examples

Multiplication rule

A consequence of the conditional probability formula is that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \mathbb{P}(B) \quad (29)$$

Further generalization:

$$\mathbb{P}\left(\bigcap_{i=1}^n A_i\right) = \mathbb{P}(A_1) \mathbb{P}(A_2|A_1) \mathbb{P}(A_3|A_1 \cap A_2) \dots \mathbb{P}\left(A_n | \bigcap_{i=1}^{n-1} A_i\right) \quad (30)$$

Example

- ④ Draw three cards from an ordinary 52-card deck without replacement. What is the probability that none is a ♠?
- ② We divide a class of 16 students — 4 male, 12 female — into 4 groups of 4 students each. What is the probability that each group has exactly one male student?

Example



Label the four students 1, 2, 3, 4. Suppose we assign them sequentially.

$A_1 = \{\text{student 2 not in the same group as student 1}\}$

$A_2 = \{\text{student 3 not in the same group as student 1 or 2}\}$

$A_3 = \{\text{student 4 not in the same group as student 1 or 2 or 3}\}$

$$\mathbb{P}(A_1) = \frac{12}{15}$$

$$\mathbb{P}(A_2 | A_1) = \frac{8}{14}$$

$$\mathbb{P}(A_3 | A_1 \cap A_2) = \frac{4}{13}$$

Therefore, using the multiplicative rule,

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \mathbb{P}(A_2 | A_1) \mathbb{P}(A_3 | A_1 \cap A_2) = \frac{12}{15} \cdot \frac{8}{14} \cdot \frac{4}{13}$$

Example

Define

$$A_i = \{\text{ith card not a } \spadesuit\} \quad i = 1, 2, 3$$

and we know:

$$\mathbb{P}(A_1) = \frac{39}{52}$$

$$\mathbb{P}(A_2 | A_1) = \frac{38}{51}$$

$$\mathbb{P}(A_3 | A_1 \cap A_2) = \frac{37}{50}$$

Therefore, using the multiplicative rule,

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \mathbb{P}(A_2 | A_1) \mathbb{P}(A_3 | A_1 \cap A_2) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50}$$

- ① Sets and probability models
- ② Conditional probability
- ③ Total probability and Bayes' rule
- ④ Independence
- ⑤ Counting

Total probability theorem

A “divide-and-conquer” approach: assume we partition the sample space into a *disjoint* set $\{A_1, \dots, A_n\}$ of events, so

$$\mathbb{P}(B) = \mathbb{P}(A_1 \cap B) + \dots + \mathbb{P}(A_n \cap B) \quad (31)$$

$$= \mathbb{P}(A_1) \mathbb{P}(B | A_1) + \dots + \mathbb{P}(A_n) \mathbb{P}(B | A_n) \quad (32)$$

This is useful when the “event” can occur due to a number of possible ways.

Example

Let U_i and D_i be the event that the switch is on or off, respectively, after the i th minute. Note it was on at time zero.

- At time 1, $\mathbb{P}(U_1) = 0.8$ and $\mathbb{P}(D_1) = 0.2$.
- Then, at time 2,

$$\begin{aligned} \mathbb{P}(U_2) &= \mathbb{P}(U_1) \mathbb{P}(U_2 | U_1) + \mathbb{P}(D_1) \mathbb{P}(U_2 | D_1) \\ &= 0.8 \cdot 0.8 + 0.2 \cdot 0.4 = 0.72 \end{aligned}$$

$$\begin{aligned} \mathbb{P}(D_2) &= \mathbb{P}(U_1) \mathbb{P}(D_2 | U_1) + \mathbb{P}(D_1) \mathbb{P}(D_2 | D_1) \\ &= 0.8 \cdot 0.2 + 0.2 \cdot 0.6 = 0.28 \end{aligned}$$

- Then, at time 3,

$$\begin{aligned} \mathbb{P}(U_3) &= \mathbb{P}(U_2) \mathbb{P}(U_3 | U_2) + \mathbb{P}(D_2) \mathbb{P}(U_3 | D_2) \\ &= 0.72 \cdot 0.8 + 0.28 \cdot 0.4 = 0.688 \end{aligned}$$

$$\begin{aligned} \mathbb{P}(D_3) &= \mathbb{P}(U_2) \mathbb{P}(D_3 | U_2) + \mathbb{P}(D_2) \mathbb{P}(D_3 | D_2) \\ &= 0.72 \cdot 0.2 + 0.28 \cdot 0.6 = 0.312 \end{aligned}$$

Example

A certain switch may be either on or off, and at the end of every 60 seconds, it may stay the same or flip. If it is originally on, there is a 0.8 probability to remain on. If it is originally off, there is a 0.6 probability to remain off. Assume at time zero it was on. What is the probability that it is on after 3 minutes?

Bayes' rule

Assume we partition the sample space into a *disjoint* set $\{A_1, \dots, A_n\}$ of events, so

$$\mathbb{P}(A_i | B) = \frac{\mathbb{P}(A_i) \mathbb{P}(B | A_i)}{\mathbb{P}(B)} \quad (33)$$

$$= \frac{\mathbb{P}(A_i) \mathbb{P}(B | A_i)}{\mathbb{P}(A_1) \mathbb{P}(B | A_1) + \dots + \mathbb{P}(A_n) \mathbb{P}(B | A_n)} \quad (34)$$

- 1 For **inference** or **inverse problem**: from *output* to *input*
- 2 $\mathbb{P}(A_i)$ is **prior probability**
- 3 $\mathbb{P}(A_i | B)$ is **posterior probability**

Bayes' rule is an extremely important tool for us!

Example

- ④ A person is infected with one of three possible diseases, which cause identical symptoms. The probability of it being disease A is 0.5, B is 0.25, and C is also 0.25. A certain drug is effective against A with probability 0.3, against B with probability 0.4, and against C with probability 0.5. The drug eventually is shown to work. What is the probability that it was disease A?
- ② A test for a disease is correct 95% of the time. Probability of a random person having that disease is 0.001. If one is just tested positive, what is the probability of having that disease?

Example

Let A = has disease, and B = tested positive. We are interested in $\mathbb{P}(A | B)$.

We know:

$$\begin{aligned}\mathbb{P}(A) &= 0.001 & \mathbb{P}(B | A) &= 0.95 \\ \mathbb{P}(A^c) &= 0.999 & \mathbb{P}(B | A^c) &= 0.05\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}(A | B) &= \frac{\mathbb{P}(A) \mathbb{P}(B | A)}{\mathbb{P}(A) \mathbb{P}(B | A) + \mathbb{P}(A^c) \mathbb{P}(B | A^c)} \\ &= \frac{0.001 \cdot 0.95}{0.001 \cdot 0.95 + 0.999 \cdot 0.05} = 0.0187\end{aligned}$$

Example

Let E denote effectiveness. Then,

$$\mathbb{P}(A) = 0.5, \quad \mathbb{P}(B) = 0.25, \quad \mathbb{P}(C) = 0.25,$$

and

$$\mathbb{P}(E | A) = 0.3, \quad \mathbb{P}(E | B) = 0.4, \quad \mathbb{P}(E | C) = 0.5.$$

We can then directly apply Bayes' rule,

$$\begin{aligned}\mathbb{P}(A | E) &= \frac{\mathbb{P}(A) \mathbb{P}(E | A)}{\mathbb{P}(A) \mathbb{P}(E | A) + \mathbb{P}(B) \mathbb{P}(E | B) + \mathbb{P}(C) \mathbb{P}(E | C)} \\ &= \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5} = 0.4\end{aligned}$$

- ① Sets and probability models
- ② Conditional probability
- ③ Total probability and Bayes' rule
- ④ Independence
- ⑤ Counting

Definition

Independence of two events A and B means

$$\mathbb{P}(A|B) = \mathbb{P}(A) \quad (35)$$

as well as

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B) \quad (36)$$

- Literally: **Knowing B gives no information about A**
- They are equivalent (assuming $\mathbb{P}(B) \neq 0$) since

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

- Relationship is symmetric:

A is independent of $B \Leftrightarrow B$ is independent of A

Example

- 1 We know $\mathbb{P}(A) = \frac{1}{6}$, $\mathbb{P}(B) = \frac{6}{36}$, and $\mathbb{P}(A \cap B) = \frac{1}{36}$. So they are independent.
- 2 We know $\mathbb{P}(C) = \frac{1}{6}$, $\mathbb{P}(D) = \frac{5}{36}$, and $\mathbb{P}(C \cap D) = \frac{1}{36}$. So they are not independent.
- 3 We know $\mathbb{P}(E) = \frac{7}{36}$, $\mathbb{P}(F) = \frac{9}{36}$, and $\mathbb{P}(E \cap F) = \frac{2}{36}$. So they are not independent.

Example

We roll two dice.

- 1 Are these two events independent?

$$A = \{\text{1st roll} = 1\} \quad B = \{\text{sum} = 7\}$$

- 2 Are these two events independent?

$$C = \{\text{1st roll} = 1\} \quad D = \{\text{sum} = 6\}$$

- 3 Are these two events independent?

$$E = \{\text{maximum of the two} = 4\} \quad F = \{\text{minimum of the two} = 2\}$$

Definition

Two events A and B can have **conditional independence** given event C :

$$\mathbb{P}(A \cap B | C) = \mathbb{P}(A | C) \mathbb{P}(B | C) \quad (37)$$

or alternatively,

$$\mathbb{P}(A | B \cap C) = \mathbb{P}(A | C) \quad (38)$$

- Literally: *When C is known, then knowing B gives no information about A*
- They are equivalent (assuming $\mathbb{P}(B | C) \neq 0$)
- Relationship is symmetric:

$A | C$ is independent of $B | C \Leftrightarrow B | C$ is independent of $A | C$

Independence vs conditional independence

A and B are independent **does not imply** they are conditional independent, **and vice versa**

- ① We have two coin tosses.

$A = \{\text{1st toss is head}\}$

$B = \{\text{2nd toss is head}\}$

$C = \{\text{two tosses have different results}\}$

Are A and B independent? How about $A|C$ and $B|C$?

- ② We have a blue coin (B) and a red coin (R), equally likely to be chosen and then tossed twice, and that $\mathbb{P}(H|B) = 0.99$ and $\mathbb{P}(H|R) = 0.01$. Let

$D_i = \{\text{ith toss is H}\}.$

Are D_1 and D_2 independent? How about $D_1|B$ and $D_2|B$?

Three of more events

When we have three events A_1 , A_2 , and A_3 , **independence** means satisfying all four conditions:

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1) \mathbb{P}(A_2) \quad (39)$$

$$\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2) \mathbb{P}(A_3) \quad (40)$$

$$\mathbb{P}(A_3 \cap A_1) = \mathbb{P}(A_3) \mathbb{P}(A_1) \quad (41)$$

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \mathbb{P}(A_2) \mathbb{P}(A_3) \quad (42)$$

Eqs. (39)–(41) stipulate **pairwise independence**.

More generally, for events $A_1, A_2, \dots, A_n$, they are independent if

$$\mathbb{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbb{P}(A_i) \quad \text{for every subset } S \text{ of } \{1, 2, \dots, n\} \quad (43)$$

Example

- ① We can check

$$\mathbb{P}(A) = \frac{1}{2}$$

$$\mathbb{P}(B) = \frac{1}{2}$$

$$\mathbb{P}(A \cap B) = \frac{1}{4}$$

$$\mathbb{P}(A|C) = \frac{1}{2}$$

$$\mathbb{P}(B|C) = \frac{1}{2}$$

$$\mathbb{P}(A \cap B|C) = 0$$

So A and B are independent, but $A|C$ and $B|C$ are not.

- ② We can compute, for $i = 1, 2$,

$$\mathbb{P}(D_i) = \mathbb{P}(B) \mathbb{P}(D_i|B) + \mathbb{P}(B^c) \mathbb{P}(D_i|B^c)$$

$$= \frac{1}{2} \cdot 0.99 + \frac{1}{2} \cdot 0.01 = \frac{1}{2}$$

$$\mathbb{P}(D_1 \cap D_2) = \mathbb{P}(B) \mathbb{P}(D_1 \cap D_2|B) + \mathbb{P}(B^c) \mathbb{P}(D_1 \cap D_2|B^c)$$

$$= \frac{1}{2} \cdot 0.99 \cdot 0.99 + \frac{1}{2} \cdot 0.01 \cdot 0.01 = 0.4901$$

$$\mathbb{P}(D_i|B) = 0.99$$

$$\mathbb{P}(D_1 \cap D_2|B) = 0.99 \cdot 0.99$$

So D_1 and D_2 are not independent, but $D_1|B$ and $D_2|B$ are.

Example

Examples:

- ① We have two coin tosses.

$A = \{\text{1st toss is head}\}$

$B = \{\text{2nd toss is head}\}$

$C = \{\text{two tosses have different results}\}$

Do A, B, and C satisfy Eqs. (39)–(42)? Are they independent?

- ② We have two dice rolls.

$D = \{\text{1st roll is one of } \{1, 2, 3\}\}$

$E = \{\text{1st roll is one of } \{3, 4, 5\}\}$

$F = \{\text{sum of two rolls is equal to } 9\}$

Do D, E, and F satisfy Eqs. (39)–(42)? Are they independent?

Example

- ① We can check

$$\begin{aligned}\mathbb{P}(A) &= \frac{1}{2} & \mathbb{P}(B) &= \frac{1}{2} & \mathbb{P}(A \cap B) &= \frac{1}{4} \\ \mathbb{P}(C) &= \frac{1}{2} & \mathbb{P}(A \cap C) &= \frac{1}{4} & \mathbb{P}(B \cap C) &= \frac{1}{4} \\ \mathbb{P}(A \cap B \cap C) &= 0\end{aligned}$$

So A, B, and C are pairwise independent, but fail Eq. (42).

- ② We can check

$$\begin{aligned}\mathbb{P}(D) &= \frac{1}{2} & \mathbb{P}(E) &= \frac{1}{2} & \mathbb{P}(F) &= \frac{4}{36} \\ \mathbb{P}(D \cap E) &= \frac{1}{6} & \mathbb{P}(E \cap F) &= \frac{3}{36} & \mathbb{P}(D \cap F) &= \frac{1}{36} \\ \mathbb{P}(D \cap E \cap F) &= \frac{1}{36}\end{aligned}$$

So D, E, and F satisfy Eq. (42), but fail pairwise independence.

Example

ANS: Let $X_i \in \{1, \dots, 1000000\}$ be the starting location of the i th read, where $i = 1, \dots, 100000$.

- ① By symmetry, since the sequence is circular, we can assume X_1 to be any number, so let's pick $X_1 = 100$. Then we need $X_2 \in [1, 199]$. So probability is $199/1000000$.
- ② Now assume the specific nucleotide is located at position 100, and we are interested in the event “ $X_i \leq 100$ for some i ”.

$$\mathbb{P}(X_i \leq 100 \text{ for some } i) = 1 - \left(\frac{1000000 - 100}{1000000} \right)^{100000} \approx 1 - e^{-10}$$

since $e^x = \lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n$, with $x = -10$.

- ③ Now assume the gene of interest is at location 1–1000, so the event is “ $X_i \leq 901$ for some i ”. Similar to the above,

$$\mathbb{P}(X_i \leq 901 \text{ for some } i) = 1 - \left(\frac{1000000 - 901}{1000000} \right)^{100000} \approx 1 - e^{-90.1}$$

Example

The DNA of a bacteria is a length 1 million circular sequence containing the nucleotides A, G, C and T. We sequence the DNA by sampling 100,000 reads randomly from the DNA, where each read is a length 100 substring of the genome. We are interested in particular in a specific gene, which is a specific length 1000 substring of the DNA.

- ① What is the probability that the first two reads overlap?
- ② What is the probability that a specific nucleotide of the DNA is covered by at least one read?
- ③ What is the probability that there is at least one read that is completely contained in the gene of interest?

- ① Sets and probability models
- ② Conditional probability
- ③ Total probability and Bayes' rule
- ④ Independence
- ⑤ Counting

Probability by counting

Many situations require us to “count” to obtain the probability:

- 1 An event A consists of a finite number of equally likely outcomes, each having probability p ,

$$\mathbb{P}(A) = p \cdot (\text{number of elements of } A)$$

- 2 There are a finite number of equally likely outcomes in A , so

$$\mathbb{P}(A) = \frac{\text{number of elements of } A}{\text{number of elements of } \Omega}$$

Some (useful) formulas

$$0! = 1$$

Stirling's approx. $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ for large n

Gamma function $n! = \Gamma(n+1)$ where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$

$$\sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} = 1$$

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

$$\sum_{k=1}^n k \binom{n}{k} = n 2^{n-1}$$

Some notations

- Factorial:

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1 \quad (44)$$

- k -permutation:

$$n(n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!} \quad (45)$$

- Combination:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (46)$$

Bernoulli trials

A sequence of independent trials with two possible outcomes, e.g., Head vs Tail, is called **Bernoulli trials**, where

$$\mathbb{P}(\text{Head}) = p \quad \mathbb{P}(\text{Tail}) = 1 - p$$

Probability of seeing k heads followed by $n - k$ tails is

$$p^k (1-p)^{n-k}$$

There are $\binom{n}{k}$ possibilities for the k heads to appear in the sequence.

Overall, the probability of having k heads in n trials is

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (47)$$

Counting outcomes

Examples:

- ① Number of possible 4-digit code:

$$10 \times 10 \times 10 \times 10 = 10^4$$

- ② Number of possible 4-digit code that cannot have duplicate:

$$10 \times 9 \times 8 \times 7 = \frac{10!}{6!}$$

- ③ Number of ways to arrange 4 digits:

$$4 \times 3 \times 2 \times 1 = 4!$$

- ④ Number of ways to pick 4 numbers without duplicate:

$$\frac{10 \times 9 \times 8 \times 7}{4!} = \binom{10}{4} = \binom{10}{6}$$

Example

How many ways to arrange the letters in HONGKONGYOU?

Ans: 11 letters mean $11!$ permutations, but there are $3!$ permutations of “O”, $2!$ permutations of “N”, and $2!$ permutations of “G”, hence

$$\frac{11!}{3! 2! 2!}$$

Ans: There are 11 positions for the letters. We partition them into a set of 3 (where we put the “O”), two sets of 2 (where we use one for “N”, one for “G”), and 4 sets of 1 (“H”, “K”, “Y”, “U”), hence

$$\binom{11}{3, 2, 2, 1, 1, 1, 1} = \frac{11!}{3! 2! 2! 1! 1! 1! 1!}$$

Partitions

Begin with a set containing n elements; divide it into r disjoint subsets, with subset i having a size of n_i , and that $\sum_{i=1}^r n_i = n$. How many ways can we do it?

Ans: First partition into two sets with n_1 elements and $n - n_1$ elements. Then, partition the latter into n_2 and $(n - n_1) - n_2$. Then, partition the latter into n_3 and $(n - n_1 - n_2) - n_3 \dots$

$$\begin{aligned} & \binom{n}{n_1} \binom{n - n_1}{n_2} \binom{n - n_1 - n_2}{n_3} \dots \binom{n - n_1 - \dots - n_{r-1}}{n_r} \\ &= \frac{n!}{n_1! (n - n_1)!} \cdot \frac{(n - n_1)!}{n_2! (n - n_1 - n_2)!} \dots \frac{(n - n_1 - \dots - n_{r-1})!}{(n - n_1 - \dots - n_r)! n_r!} \\ &= \frac{n!}{n_1! n_2! \dots n_r!} = \binom{n}{n_1, n_2, \dots, n_r} \end{aligned}$$

Example

We divide a class of 16 students — 4 male, 12 female — into 4 groups of 4 students each. What is the probability that each group has exactly one male student? (We solved before, but here we use partitions)

Ans: Together, there are $\binom{16}{4, 4, 4, 4} = \frac{16!}{4! 4! 4! 4!} = A$ partitions.

For the male, there are $\binom{4}{1, 1, 1, 1} = \frac{4!}{1! 1! 1! 1!} = B$ partitions.

For the female, there are $\binom{12}{3, 3, 3, 3} = \frac{12!}{3! 3! 3! 3!} = C$ partitions.

Overall probability = $\frac{B \cdot C}{A} = \frac{4! 12!}{3! 3! 3! 3!} \cdot \frac{4! 4! 4! 4!}{16!} = \frac{12 \cdot 8 \cdot 4}{15 \cdot 14 \cdot 13}$

Summary

We have outlined the basic mathematical foundation for calculating probabilities. Next, we will concentrate our attention to assigning numerical values to different outcomes, and proceed with developing tools to compute and interpret those values. We will divide broadly into two situations:

- Discrete probability
- Continuous probability

and introduce the notion of *random variables*.